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HINTS

FOR THE

SOLUTION OF PROBLEMS

IN THE THIRD EDITION OF

SOLID GEOMETRY

BY

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PREFACE.

THE fulfilment of my promise to give an appendix, containing solutions or hints for the solution of all the problems given in my Third Edition of *Solid Geometry*, has entailed much labour; but this labour will not have been thrown away if it should in any degree have added to the usefulness of the book; at all events it has enabled me to detect many errors and omissions in the statement of the problems which might have given trouble to the student. A table of these errata is given on the following page.

Mr. Chree, Mr. Berry, and Mr. Richmond have shewn no discontinuity in their kindness, for they have not only corrected the proof sheets, but have detected important errors in the problems, as *e.g.* in LX. (7) (Mr. Berry), and in LVIII. (3) (Mr. Richmond); the geometrical solutions of LII. (1) and LXIV. (9) were given by Mr. Berry and Mr. Richmond. I wish to thank Mr. Chree especially for his superintendence of the printing during my absence in the Long Vacation, and I am glad to have this opportunity of noticing a great improvement on the last two lines of my solution of XLIII. (4), which was suggested by him but unfortunately arrived too late, viz. "if (x, y, z, w) be the centre, the left side of $(1) = -R^2$."

ERRATA IN HINTS FOR SOLUTION.

PAGE
44, XXXI. (9), reference to fig. 1 is omitted.
75, L. (2), line 2, *for* $(r\xi^2$, *read* $\frac{1}{2}(r\xi^2$.
77, LII. (6), line 2, *for* $x^{-2}y^{-2}$, *read* $x^{-1}y^{-1}$.
79, LII. (2), line 4, *for* PS, *read* QS.

PROBLEMS.

ERRATA *majora.*

PAGE

113, XX. (8), *insert* $+abc=0$ *after* cxy.

128, XXIII. (9), line 5, *insert* $-$ *before* x.

179, XXIX. (1), line 12, *for* $\sqrt{\tfrac{3}{2}}$, *read* $\sqrt{3}$.
line 13, *for* 1, 0, 1, *read* 1, 0, -1.

180, XXIX. (9), line 3, *insert* $+a''\sqrt{a}/(a+b)$ *after* $y\sqrt{b}$.

225, XXXVI. (9), *add* for the same height of the luminous point.
XXXVII. (7), line 2, *dele* double.

226, XXXVIII. (8), *add* a is the intersection of tangent planes at B, C, D.

236, XL. (3), *dele* of revolution.
(7), *insert* $+a'$ *after* $-C'z$).

301, XLIX. (6), *for* $4\pi\{1-c/\sqrt{(a^2+c^2)}\}$, *read* $4\pi a/\sqrt{(a^2+c^2)}$.

303, LI. (7), line 3, *for* the portion, *read* any portion.
line 4, *for* π, *read* 2π.
line 5, *add* estimated symmetrically with respect to the portion.
(9), line 4, *for* ; also &c., *read* along circular parts of their intersection.

328, LV. (3), *add* and the central circular sections.
(5), *for* conoidal surface, *read* right conoid.

329, LVI. (2), line 6, *for* tangent...at P, *read* generator of the scroll through P.

354, LVII. (7), *for* $\frac{dq}{dp}$, *read* $\left(\frac{dq}{dp}\right)^2$.
LVIII. (3), *add* if p, q be measured along fixed generating lines.
(4), line 6, *for* conicoid, *read* helicoidal surface.

356, LX. (2), line 5, *insert* $-$ *before* $\rho_r\sigma_p\tau_q$.
(6), *for* epicycloid, *read* hypocycloid.
(7), line 6, *insert* $+i\{\phi(p+iu)-\phi(p-iu)\}$ *after* $f(p-iu)$.

372, LXII. (1), line 6, *for* m^2, *read* m.

389, LXVI. (5), *for* $n-2$, *read* $2(n-2)$.

ERRATA *minora.*

89, XV. (4), *for* BC, *read* PC.

101, XVIII. (14), line 1, *for* (11), *read* (14).

126, XXI. (10), *for* b^2-m^2, *read* $(b^2-m^2)^2$.

181, XXXI. (7), line 4, *for* $a'^2b'^2c'^2$, *read* $27a'^2b'^2c'^2$.

224, XXXV. (6), *for* ax, *read* az.

248, XLII. (8), line 5, *add* and abc *after* $a'b'c$.

249, XLIII. (10), *for* pair, *read* pairs.

276, XLVI. (7), *for* β, *read* γ.

302, L. (4), *for* p/z, *read* p/x.

329, LVI. (4), *for* ϕ, *read* ψ.

354, LVIII. (1), *for* square, *read* rectangular.

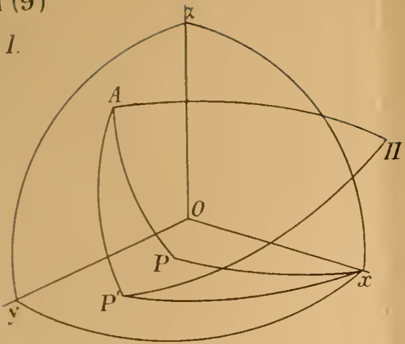
356, LX. (7), line 2, *for* α, *read* a.
lines 9, 10, 11, *for* index ^{2}, *read* ^{3}.
line 10, *omit* $-$ *before* $\sin sp$.
line 11, *for* $\sin sq$, *read* $\sin sp$.

389, LXVI. (3), line 6, *for* a'^2, *read* a^2.

402, LXVII. (6), *for* $(5a^{-1}+b^{-1})$, *read* $(5a+b)$.

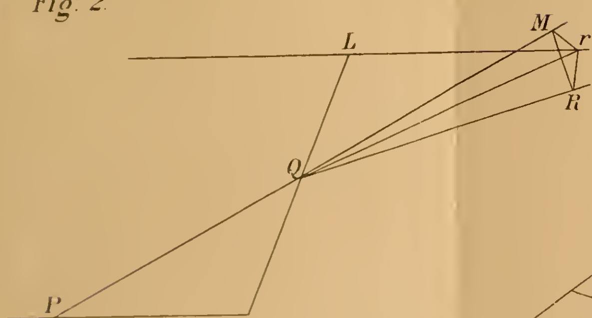
XXXI (9)

Fig. 1.



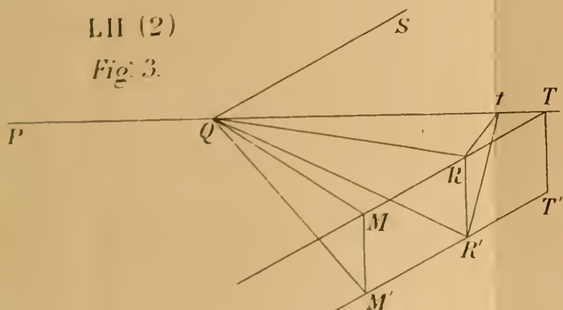
XLVI (1)

Fig. 2.



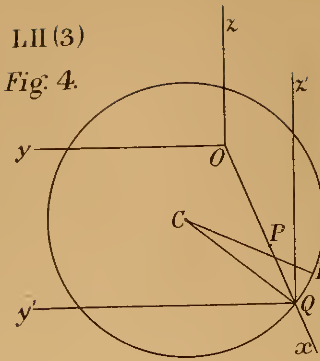
LII (2)

Fig. 3.



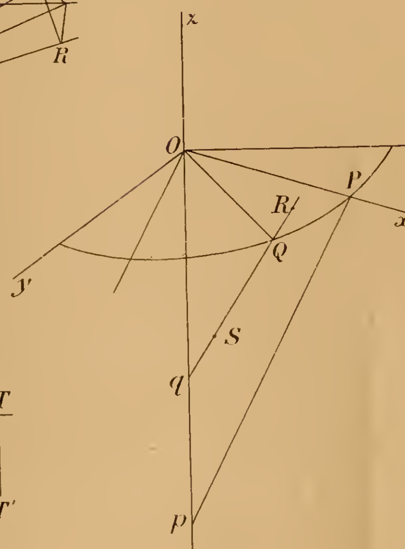
LII (3)

Fig. 4.



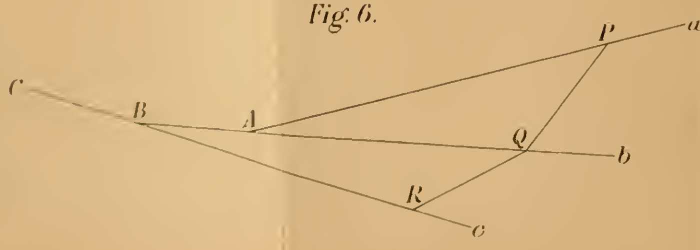
LII (4)

Fig. 5.



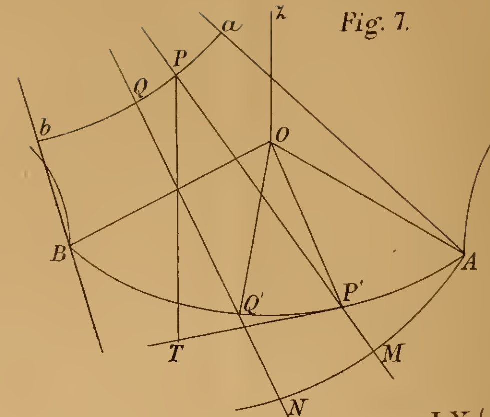
LIII (6)

Fig. 6.



LIX (1)

Fig. 7.

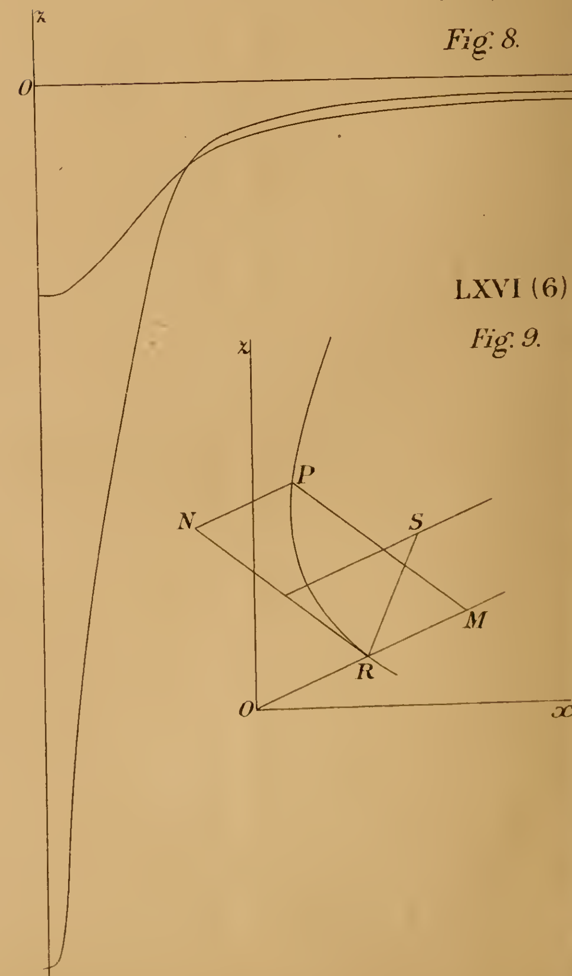


LX (5)

Fig. 8.

LXVI (6)

Fig. 9.



HINTS FOR THE SOLUTION OF PROBLEMS

IN THE THIRD EDITION OF

FROST'S SOLID GEOMETRY.

I.

(1) Two points $(\frac{5}{2}a, \frac{3}{2}a, \pm 2a)$.

(2) Prove that $(x-y)^2=0$, two pairs of coincident points (a, a, a) $(-a, -a, -a)$.

(5) Circle in the plane xy.

II.

(1) (i) Cylinder on a circular base touching Oy. (ii) Traces on zx, zy, parabolas; the section by any plane parallel to xy is a straight line. (iii) Sphere, whose centre is (a, b, c). (iv) Generated by parabolas revolving round Oz; or by circles, centres in Oz, intersecting parabolic traces on xz, yz. (v) Planes $z=\pm h$ cut the surface in straight lines through Oz inclined to the plane zx at an angle $\tan^{-1}(h^2/c^2)$. (vi) Generated by a hyperbola parallel to plane xy. (vii) Generated by an ellipse, one axis constant, the other changing from 0 to ∞. (viii) Trace on plane yz the parabola $y^2=cz$, and on plane $z=h$ an equal parabola with vertex $(h, -h, h)$, generating a parabolic cylinder.

(2) Fig. page 3, (i) $r=a\sin\theta$ gives a circle in plane POM, touching Oz, the same for all values of ϕ. (ii) If a circle in plane xy touch Oy and pass through M, $r=OM$ for all values of θ, giving a circle in plane POM. (iii) $\theta=\frac{1}{2}\pi+\frac{1}{4}\pi\sin 4\phi$ for all values of r, OP makes $\angle\,\frac{1}{4}\pi\sin 4\phi$ below plane xy, and generates a surface cutting xy where $\phi=0, \frac{1}{4}\pi, \frac{1}{2}\pi$, &c.

III.

(1) Art. 23, let l, m, n be the direction-cosines, $l\cos\alpha + m\sin\alpha = 0$, $m\sin\gamma + n\cos\gamma = 0$.

(2) $ll' + mm' + nn' = \frac{1}{2}$, $\therefore (l - l')^2 + \ldots = 1$, $l(l - l') + \ldots = \frac{1}{2}$, $l(l - l') + \ldots = -\frac{1}{2}$.

(3) $\sin^2\alpha + \sin^2(\alpha + 45°) + \sin^2(\alpha + 90°) = 1$, $\therefore \alpha + 45° = 0$. Also $\cos^2\alpha + \cos^2 2\alpha + \cos^2 3\alpha = 1$, $\therefore \cos 2\alpha \cos 3\alpha \cos\alpha = 0$.

(4) Fig. page 44, AE, BE perpendicular to CD. $\cos AEB = (2AE^2 - AB^2)/2AE^2$, and $AE = \frac{1}{2}\sqrt{3}\,AB$.

(5) $2(l^2 + m^2) - (l + m)^2 = n^2 = (l - m)^2$, the direction-cosines are $0, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}$, and $\sqrt{\frac{1}{2}}, 0, -\sqrt{\frac{1}{2}}$.

(6) Elementary sector of the circular base = elementary triangle of surface $\times a/l$.

(7) The relation is not altered when $-l$ is written for l, or $-m$ for m, or $-n$ for n.

(8) Areas are as $3 : 4 : 5$, and, by Art. 36, $\lambda : \mu : \nu = 3 : 4 : 5$, $\therefore \nu = \sqrt{\frac{1}{2}}$.

IV.

(1) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\therefore \cos(\alpha + \beta)\cos(\alpha - \beta) + \cos^2\gamma = 0$, whatever γ is, $\alpha + \beta$ is least when $\alpha = \beta$, similarly $\alpha = \gamma$, $\therefore 3\cos^2\alpha = 1$, whence 3α, the least possible value of $\alpha + \beta + \gamma$, is found.

(2) Shew that $\lambda l + \mu m + \nu n = 0$, and $\lambda + \mu + \nu = 0$; λ, μ, ν being the direction-cosines.

(3) Shew that $c^2(\alpha l^2 + \beta m^2) + \gamma(al + bm)^2 = 0$, hence that

$$l_1 l_2 : m_1 m_2 : n_1 n_2 = c^2\beta + b^2\gamma : a^2\gamma + c^2\alpha : b^2\alpha + a^2\beta.$$

Condition of parallelism is that of equal roots of the quadratic in l, m.

(4) Similarly.

(5) λ, μ, ν direction-cosines required, $\lambda l + \ldots = \lambda l' + \ldots = \lambda l'' + \ldots = \rho$,

$$\therefore \lambda \times \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = \rho \begin{vmatrix} 1, & m, & n \\ 1, & m', & n' \\ 1, & m'', & n'' \end{vmatrix}$$

$$\text{and } \lambda : \mu : \nu = \begin{vmatrix} 1, & m, & n \\ 1, & m', & n' \\ 1, & m'', & n'' \end{vmatrix} : \ldots : \ldots.$$

For the second case, shew that

$$l = m'n'' - m''n', \quad \therefore \lambda : \mu : \nu = l + l' + l'' : \ldots : \ldots.$$

(6) Art. 26, $u + v + w = 2a\beta\gamma + 2b\gamma\alpha + 2c\alpha\beta$, $P = a^2\alpha^2 + \ldots - 2bc\beta\gamma \ldots$.

(7) l, m, n and l', m', n' direction cosines of the lines. Prove $l - l' = -(m - m')$ and $l + l' = m + m'$, $\therefore\ l = m'$, $m = l'$, $2lm = -n^2$, $(l - m)^2 = 3n^2$, then $l = \frac{1}{2}(-1 \pm \sqrt{\frac{1}{3}})$, $m = \frac{1}{2}(1 \pm \sqrt{\frac{1}{3}})$, $n = \mp\sqrt{\frac{1}{3}}$.

(8) Art. 25, λ', μ', ν' required direction cosines,

$$\lambda + \lambda' = 2\sqrt{\tfrac{1}{3}}\,(\lambda + \mu + \nu)\,\sqrt{\tfrac{1}{3}}.$$

(9) $1 - \frac{1}{2}(\delta\theta)^2 + \ldots = l\,(l + \delta l) + \ldots$ and $1 = (l + \delta l)^2 + \ldots$.

(10) Art. 36, a an edge of the cube, $\Sigma A_1 = a^2 = \Sigma A_3 = \Sigma A_2$, normal to plane of maximum projection has equal direction cosines and maximum area $= a^2\sqrt{3}$.

(11) Let θ be the inclination of the planes; the perpendicular from D' on the plane $ABC = DD'\cos\theta$.

(12) Art. 28, L, M, N direction-ratios,

$$-l + L + M\cos\nu + N\cos\mu = 0,$$

and three other equations; eliminate L, M and N.

V.

(1) $x = z = -\frac{1}{2}y$, $\cos\alpha = \cos\gamma = -\frac{1}{2}\cos\beta$, $\sec\beta = -\sqrt{\frac{3}{2}}$, $\sec 2\beta = 3$.

(2) Use the three equations

$$(x - 1)(y - 1)(x - y) = 0, \quad (y - 1)(z - 1)(y - z) = 0,$$
$$(z - 1)(x - 1)(z - x) = 0,$$

satisfied by $x = y = 1$, $x = 1$, $y = z$, &c., and $x = y = z$; straight lines passing through $(1, 1, 1)$.

(3) Satisfied by $x = y = z$, direction cosines $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{3}}$.

(4) $x^2 + 2xz = 0$, $\therefore$ straight lines are $x = 0$, $y = 0$ and $x = y = -2z$.

(5) Straight line is $(x - b)/(c - b) = (y - c)/(a - c) = (z - a)/(b - a)$ perpendicular to straight line $2x/(b + c) = 2y/(c + a) = 2x/(a + b)$ and the other two lines.

(6) It is the distance from the origin to the projection of the line on the plane xy, and is $(am \sim bl)/\sqrt{(l^2 + m^2)}$. The equations are $lx + my = 0$ and $z = \gamma$, which meets the given line, shew that $\gamma\,(l^2 + m^2) + n\,(al + bm) = 0$.

(7) Line joining centres of the edge and diagonal is the shortest distance $= a/\sqrt{2}$.

(8) Take $y = A + Bz/c$ and $mx = A' + B'z/c$ for the intersecting line. Shew that $A' = B$ and $B' = A$, and make $A = m\lambda\sin\theta$, $B = \lambda\cos\theta$.

(9) The points are $(a\cos\alpha,\ a\sin\alpha,\ c)$; $(\pm b\cos\alpha,\ \mp b\sin\alpha,\ -c)$.

(10) Art. 59, cylinder of evanescent radius. Equation may be written $(ny - mz)^2 + (lz - nx)^2 + (mx - ly)^2 = 0$.

(11) Art. 64, for the locus, $z = \frac{1}{2}(c - c) = 0$.

VI.

(1) For the given line $mx+ny+lz=a$ and $nx+ly+mz=0$, $\therefore x:y:z=l^2-mn:m^2-nl:n^2-lm=L:M:N$. If λ, μ, ν be direction-cosines of the required line, $L\lambda+M\mu+N\nu=0$, and $n\lambda+l\mu+m\nu=0$. At the point of intersection $x=\lambda r$, &c., and $(m\lambda+n\mu+l\nu)\,r=a$; shew that $r\surd(L^2+M^2+N^2)=a\surd(l^2+m^2+n^2)$.

(2) The straight lines must satisfy the three equations

$$(x+1)(y+1)(x-y)(x+y-1)=0,\quad (y+1)(z+1)(y-z)(y+z-1)=0,$$
$$\text{and}\quad (z+1)(x+1)(z-x)(z+x-1)=0,$$

their equations are of the five types $x+1=0$, $z+1=0$ (i), $x+1=0$, $y=z$ (ii), $x+1=0$, $y+z=1$ (iii), $x=y$, $y+z=1$ (iv), three of each, and $x=y=z$ (v); (iv) and (v) are four diagonals of a cube.

(3) Eliminate z, and shew that $x_1x_2 : y_1y_2 : z_1z_2 = b-c : c-a : a-b$.

(4) Shew that $mc-nb-(l^2+m^2+n^2)\,x+l\,(lx+my+nz)=0$, l, m, n are direction-cosines.

(5) Shew that $\nu b-\mu c+(l\lambda+m\mu+n\nu)\,x-l\,(\lambda x+\mu y+\nu z)=0$.

(6) Art. 64, direction cosines of $A'C$ are as

$$(BC+A'B')\cos\alpha : (BC-A'B')\sin\alpha : BB',$$

those of $B'A'$ are as $\cos\alpha : -\sin\alpha : 0$; $\therefore BC\cos 2\alpha=-A'B'$. Similarly $B'C'\cos 2\alpha=-AB$.

(7) $\pm\alpha$ the inclination of the rays to Ox in plane zx, β that of the straight line in plane xy of the mirror. Shew that the cosine of the angles between the rays and line is $\cos\alpha\cos\beta$ for each ray. Geometrically, the incident ray, and reflected ray produced backwards, are similarly placed with respect to the line.

(8) $x_1y_1z_1$, $x_2y_2z_2$ proportional to the direction-cosines of the two lines, eliminate z and obtain

$$x_1x_2 : y_1y_2 : x_1y_2+x_2y_1 = cm^2+bn^2 : an^2+cl^2 : -2clm.$$

Deduce the value of $x_1x_2+y_1y_2+z_1z_2 : \surd\{(x_1y_2-x_2y_1)^2+\ldots\}=\cos\theta:\sin\theta$.

(9) The projections of the straight lines on the plane xy will form a harmonic pencil. The equations of the projections are $w^2(ax^2+by^2)+c\,(ux+vy)^2=0$ and $w^2(Ax^2+By^2)+C\,(ux+vy)^2=0$, when y is given, let x_1x_2, X_1, X_2 be the roots of the equations $X_1-x_1 : X_2-x_1 = x_2-X_1 : X_2-x_2$, whence

$$2x_1x_2+2X_1X_2-(x_1+x_2)(X_1+X_2)=0.$$

(10) Axes as in Art. 64, r, r' distances of points on the two lines from the shortest line, (x, y, z) the middle point,

$$2x=(r+r')\cos\alpha,\quad 2y=(r-r')\sin\alpha,\quad z=0,$$
$$(r-r')^2\cos^2\alpha+(r+r')^2\sin^2\alpha=\text{constant};$$

$\therefore$ locus is an ellipse, $y^2\cos^4\alpha+x^2\sin^4\alpha=\text{constant}$.

(11) Sphere, centre O, cuts the axes in X, Y, Z; draw ZU perpendicular to the side XY of the spherical triangle XYZ; let $XU=\delta$, $ZU=\psi$, triangles ZUX, ZUY are right-angled,

$$\therefore \cos\beta = \cos\psi \cos\delta, \quad \cos\alpha = \cos\psi \cos(\gamma-\delta),$$

prove that $\sin^2\psi \sin^2\gamma = 1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma$, $z \sin\psi$ is the distance of a point from plane xy.

(12) $\delta\lambda$, $\delta\mu$, $\delta\lambda'$, $\delta\mu'$ are increments of λ, μ, λ', μ' when the parameter has an infinitesimal change, at the point of intersection $x\delta\lambda+\delta\mu=0$ and $x\delta\lambda'+\delta\mu'=0$, $\therefore \delta\lambda . \delta\mu' = \delta\lambda' . \delta\mu$.

VII.

(1) Equation of the plane being $\lambda(x-a)+\mu(y-b)+\nu(z-c)=0$, $\lambda l+\mu m+\nu n=0$ and $\lambda a+\mu b+\nu c=0$, whence λ, μ and ν; given the two lines $(x-a)/l=\ldots$ and $(x-a')/l'=\ldots$, the equations of the required line are $x(bn-cm)+\ldots=0$ (1) and $x(b'n'-c'm')+\ldots=0$ (2). When $l'=l$, &c., both equations are satisfied by $x/l=y/m=z/n$, parallel to the given lines, but the required line will be indeterminate if (1) and (2) be coincident, the condition being

$$bn-cm : b'n-c'm = cl-an : c'l-a'n.$$

(2) Shew that the equation of the plane is $x+y+z=a+b+c$ (1). Take $\lambda(x-a)+\ldots=0$ for the plane through (a, b, c) and (b, c, a). Prove that $\lambda : \mu : \nu = a+b-2c : b+c-2a : c+a-2b$, and that its intersection with (1) is $x/(a-b)=y/(b-c)=z/(c-a)$, another of the intersections is $x/(b-c)=y/(c-a)=z/(a-b)$, inclined at an angle α to the former, where

$$\pm\cos\alpha = \{(a-b)(b-c)+\ldots\}/\{(a-b)^2+(b-c)^2+(c-a)^2\} = \tfrac{1}{2}.$$

(3) $x+y+z-3=0$ and $x+y-3z+1=0$, for the plane through the origin $4(x+y)-8z=0$.

(4) Common point is (1, 3, 2). Plane equally inclined to the axes is $x-1+y-3+z-2=0$.

(5) By Art. 37, for the dividing point

$$x = \{\lambda(a+lr)+\lambda'(a'+l'r')\}/(\lambda+\lambda'), \quad y=\ldots\ z=\ldots,$$

eliminate r and r'.

(6) Any point in the line joining (x, y, z) (x', y', z') satisfies the equation $A(\lambda x+\lambda' x')+B(\lambda y+\lambda' y')+C(\lambda z+\lambda' z')+D(\lambda+\lambda')=0$ for all values of $\lambda : \lambda'$.

(7) Where the plane $x/a'+\ldots=1$ meets the diagonal $x/a=y/b=z/c=s$, $(a/a'+b/b'+c/c')s=1$ and $s=AB'/AB$.

(8) P, P' the points (a, b, c), (a', b', c'), the two straight lines are PQ parallel to OP', and $P'Q$ to OP; $OPQP'$ is the plane.

(9) Adding the first two equations gives the third.

(10) All three are perpendicular to the same line.

(11) l', m', n' direction-cosine of a second line, $\therefore \lambda l' + \mu m' + \nu n' = 0$.

(12) The two planes are $\lambda x + \mu (y + 2a) + 6\nu (z - a) = 0$ and $\lambda' (x + a) + 2\mu' y - 12\nu' z$. For the first of these $\lambda + \mu + \nu = 0$, $\lambda + \frac{1}{2}\mu - \frac{1}{2}\nu = 0$, $\therefore \lambda : \mu : \nu = 2 : -3 : 1$, and the equation becomes $2x - 3(y + 2a) + 6(z - a) = 0$, the perpendicular from O on this plane is $\frac{12}{7}a$; similarly for the second the perpendicular from O is $-\frac{6}{7}a$.

(13) A, B, C, D the four points, for BCD, $2x + y + z = 11$; for CDA, $x + y + 2z = 9$; for DAB, $z - y = 1$; for ABC, $y - x = 1$; dihedral angles containing BD and AC, $90°$; AB, $60°$; BD, CA supplementary.

(14) If ABC be the triangle, CD perpendicular to AB, AB is perpendicular to plane COD, whose equation is $\therefore ax = by$. Shew that the orthocentre is given by $ax = by = cz = (a^{-2} + b^{-2} + c^{-2})^{-1}$.

VIII.

(1) Each member $= \dfrac{\alpha x + \beta y + \gamma z}{lx + my + nz} = \dfrac{\alpha l + \beta m + \gamma n}{l^2 + m^2 + n^2}$.

(2) Any plane through the first is $\lambda (x-a) + \mu (y-b) + \nu (z-c) = 0$, if $\lambda l + \mu m + \nu n = 0$, and if it contain the second

$$\lambda (a' - a) + \mu (b' - b) + \nu (c' - c) = 0;$$

$\therefore$ plane required is $\{m (c' - c) - n (b' - b)\} (x - a) + \ldots = 0$. The condition makes the straight lines coincident and the plane indeterminate.

(3) Each member $= \dfrac{ax + by + cz}{x(m-n) + y(n-l) + z(l-m)} = \dfrac{la + mb + nc}{0}$.

(4) Plane through the first is $y/b + z/c - 1 + \lambda x = 0$, and being parallel to the second is $y/b + z/c - x/a = 1$; plane through the second parallel to the first, $x/a - z/c - y/b = 1$; take the sum of the perpendiculars from the origin.

(5) Plane through the line is $\lambda x + \mu y + \nu z = 0$, where $\lambda l + \mu m + \nu n = 0$, and

$$\lambda l' + \mu m' + \nu n' = \pm \cos\alpha \sqrt{(\lambda^2 + \mu^2 + \nu^2)} \sqrt{(l'^2 + m'^2 + n'^2)},$$

whence the equation of the two planes, since $\lambda : \mu : \nu = ny - mz : \ldots : \ldots$ As the plane turns round the line, there are two positions for every angle at which it intersects the given plane, which become coincident when the angle is equal to the angle between the given plane and line, and this is the least value of α.

To shew analytically the coincidence of the two positions; for the critical value of α, $(l^2+\dots)(l'^2+\dots)\sin^2\alpha = (ll'+\dots)^2$, prove that $(\lambda^2+\mu^2+\nu^2)(ll'+\dots)^2 = (l^2+m^2+n^2)\{(\mu n'-\nu m')^2+\dots\}$, (1) write for $\lambda^2+\mu^2+\nu^2$, $(l^2+m^2+n^2)(x^2+y^2+z^2)-(lx+my+nz)^2$ and for $\mu n'-\nu m'$, $l(l'x+m'y+n'z)-x(ll'+mm'+nn')$, whence by (1) $\{(l^2+m^2+n^2)(l'x+m'y+n'z)-(ll'+mm'+nn')(lx+my+nz)\}^2=0$, the geometrical interpretation of which is obvious.

(6) For a plane $z\cos\beta - x\sin\beta+\mu y=0$, $\cos^2\beta=(1+\mu^2)\cos^2\alpha$ eliminate μ.

(7) Take a, a', &c., for reciprocals of OA, OA', &c., the three lines being the axes. Shew that the intersections of each pair lie in the plane $(a+a')x+(b+b')y+(c+c')z=2$.

(8) As in Art. 82, since $\rho=\sigma\sqrt{3}$, the bisecting planes are $(b+c-2a)x+(c+a-2b)y+(a+b-2c)z=\pm\sqrt{3}\{(b-c)x+(c-a)y+(a-b)z\}$, and making $a+c=2b$, $3(-x+z)=\pm\sqrt{3}(x-2y+z)$, $y=0$, $(\sqrt{3}\pm 1)x=(\sqrt{3}\mp 1)z$.

(9) If the distances of (x, y, z) from the three planes be each ρ, $p_1-l_1x-m_1y-n_1z=\rho$, &c.; $\therefore l_1p_1+l_2p_2+l_3p_3-x=(l_1+l_2+l_3)\rho$, &c.

(10) Take plane xy parallel to the given plane, and planes zx, zy each to contain one of the lines; let the equations of the two lines be $x=mz+f$, $y=0$; $y=nz+g$, $x=0$; and let (ξ, η, ζ) be a point in the moving line when its distance from plane xy is ζ, dividing it in the ratio $\lambda':\lambda$,

$$\therefore\ \xi=\lambda(m\zeta+f)/(\lambda+\lambda'),\quad \eta=\lambda'(n\zeta+g)/(\lambda+\lambda').$$

(11) Equate each member to ρ, eliminate x, y, and z; $\therefore\ (a-\rho)(b-\rho)(c-\rho)-a'^2(a-\rho)-b'^2(b-\rho)-c'^2(c-\rho)+2a'b'c'=0$, giving generally three values of ρ, and therefore three lines.

Each member $=\{(aa'-b'c')x+(a'c'-bb')y\}/(a'x-b'y)=$ &c., hence, with the given conditions, either

$$a'x=b'y=c'z \quad\text{or}\quad x/a'+y/b'+z/c'=0.$$

(12) Equations of a line intersecting the first two are

$$x+y-z-1=\alpha x \quad\text{and}\quad x+y-z+1=\beta y,\quad (1)$$

but where it meets the third line, $2y=\alpha x$ and $2x=\beta y$, $\therefore\ \alpha\beta=4$, or $(x+y-z)^2-1=4xy$. Where it meets the fourth line $x-1=\alpha x$ and $x+1=-\beta x$; $\therefore\ \alpha-\beta=2$, $\therefore\ \alpha=\pm\sqrt{5}+1$, $\beta=\pm\sqrt{5}-1$. Equations (1) give $x:y:z=\beta:\alpha:(1-\alpha)\beta+\alpha$; $\therefore$ direction-cosines of the two lines are as $\pm\sqrt{5}-1 : \pm\sqrt{5}+1 : 2(\pm\sqrt{5}-2)$.

IX.

(1) Every point in either bisector is equidistant from the two given planes.

(2) For every point in the plane, $Ax=3V$ with quadriplanar, or $x=1$ with tetrahedral coordinates. Arts. 98, 99.

(3) Take P a point in AO,
vol. $PACD =$ vol. $AOCD -$ vol. $POCD$, $\therefore By = \Delta OCD(AO-x)$.

(4) For a point at an infinite distance in AB, $z=0$, $w=0$, and $x+y=0$, by Art. 110, the last being the equation of a plane through CD.

(5) Take P the given point, the plane is PAD, the line is AP.

(6) Take P, Q the first points of intersection in AB, CD,

$$x=PB/AB=\tfrac{2}{3},\ y=PA/BA=\tfrac{1}{3},\ \therefore\ x=2y.$$

(7) Centre of gravity is the same as of four equal masses placed at A, B, C, D, $\therefore$ distant from $BCD=\frac{1}{4}p_0$; $\therefore x=\frac{1}{4}=y=z=w$.

(8) The middle points of AB, CD are $(\frac{1}{2}, \frac{1}{2}, 0, 0)$, $(0, 0, \frac{1}{2}, \frac{1}{2})$, $\therefore x=y$, $z=w$ is a straight line containing both points, and also the point $x=y=z=w$, similarly for the other opposite edges.

X.

(1) Plane $lx+my+nz+rw=0$ cuts AB in P, where $lx+my=0$; $y=0$, $lx-my=0$, $x=0$, and $lx+my=0$ give four planes through CD cutting AB harmonically, the six planes all pass through the point $lx=my=nz=rw$.

(2) Equation of AO is $y/m=z/n=w/r$, coordinates of O are ls, ms, ns, rs, where $(l+m+n+r)s=1$, at A'; $x+1=2ls$, $y=2ms$, $z=2ns$, $w=2rs$.

(3) Let r_1, r_2 be the radii; the centres are $(-r_1, r_1, r_1, r_1)$ and $(r_2, -r_2, r_2, r_2)$; $\therefore$ the straight line $x+y=0$ and $z=w$ contains both. $PC:DC=w:s_0$, $CD:PD=r_0:z$, and $z=w$; $\therefore PC:PD=r_0:s_0=D:C$.

(4) Take the order of trisection $APQB$ and $CP'Q'D$; P, P' are $(\frac{2}{3}, \frac{1}{3}, 0, 0)$, $(0, 0, \frac{2}{3}, \frac{1}{3})$ middle point of PP' is $(\frac{1}{3}, \frac{1}{6}, \frac{1}{3}, \frac{1}{6})$, similarly middle point of QQ' is $(\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3})$, the straight line $x=z$, $y=w$ contains both and bisects BD and AC, similarly for the other arrangement.

(5) Centres of AB, CD are $(\frac{1}{2}, \frac{1}{2}, 0, 0)$, $(0, 0, \frac{1}{2}, \frac{1}{2})$; by Arts. 101, 103, $-l^2=\frac{1}{4}(-a^2-b^2+c^2-a'^2-b'^2+c'^2)=-\frac{1}{4}\sigma^2$; by Art. 105, $2\sigma\sigma'\cos\omega=2(a^2\sim a'^2)$, $\therefore 4ll'\cos\omega=a^2\sim a'^2$.

(6) Oa, Ob perpendicular to BCD, ACD, the distance of b from plane BCD is $Oa + Ob\cos(AB) = x + y\cos(AB)$, similarly for the feet of the perpendiculars on ABC, ABD, and the conditions of the problem give

$$x = \tfrac{1}{4}\{3x + y\cos(AB) + z\cos(AC) + w\cos(AD)\}, \quad (1)$$

$$\text{but } A = B\cos(AB) + C\cos(AC) + D\cos(AD);$$

$\therefore x/A = y/B = z/C = \omega/D$ satisfies the four equations corresponding to (1).

(7) Let ρ, ρ_1 be the radii of the two spheres touching BCD internally and externally; by the equations of Arts. 98, 99,

$$\rho/p_0 + \rho/q_0 + \rho/r_0 + \rho/s_0 = 1, \text{ and } -\rho_1/p_0 + \rho_1/q_0 + \rho_1/r_0 + \rho_1/s_0 = 1.$$

(8) Let O be the centre, and R the radius of the circumscribing sphere, and let AO produced meet plane BCD in P, the tetrahedral coordinates of O are OP/AP, &c., or $1 - R/AP$, $1 - R/BP$, ...;

$$\therefore 4 - R/AP - R/BP - R/CP - R/DP = 1.$$

(9) Take O the centre, ρ the radius of the inscribed sphere, plane $OA'B'$ cuts CD at right angles in P, $A'P\sin\gamma = \rho\cot\tfrac{1}{2}\gamma\sin\gamma$, hence quadriplanar coordinates of A' are 0, $2\rho\cos^2\tfrac{1}{2}\gamma$, $2\rho\cos^2\tfrac{1}{2}\beta$, $2\rho\cos^2\tfrac{1}{2}a$, and equations of AA' and BB' are

$$y/\cos^2\tfrac{1}{2}\gamma = z/\cos^2\tfrac{1}{2}\beta = w/\cos^2\tfrac{1}{2}a,$$

$$\text{and } x/\cos^2\tfrac{1}{2}\gamma = z/\cos^2\tfrac{1}{2}\alpha = w/\cos^2\tfrac{1}{2}b;$$

if AA' and BB' intersect $\cos^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}a = \cos^2\tfrac{1}{2}\beta\cos^2\tfrac{1}{2}b$, so also for CC', DD'.

(10) Draw Aa perpendicular to BCD, cutting the given plane in P, the distance of a from plane ACD is $p_0\cos(AB)$, $\therefore$ at a, $y = p_0\cos(AB)/q_0$, &c., and the equations of Aa are

$$\frac{yq_0}{\cos(AB)} = \frac{zr_0}{\cos(AC)} = \frac{ws_0}{\cos(AD)} = (1 - x)p_0 = AP = R;$$

at P, $p(1 - R/p_0) + R\{\cos(AB)\,q/q_0 + \cos(AC)\,r/r_0 + \cos(AD)\,w/w_0\} = 0$, and by Art. 109 or 112, $p = R\cos(p, p_0)$, $\therefore$ &c.

XI.

(1) Let a, b be the middle points of BC, CA, $q + r = 0$, $p = 0$ are the equations of a and A, $p + r = 0$, $q = 0$ those of b and B, $p + q + r = 0$ is the equation of a point in both Aa and Bb, *i.e.* of G the centre of gravity of ABC. $p + q + r + s = 0$ is a point in the line joining $p + q + r = 0$, and $s = 0$, *i.e.* in GD; similarly for the other lines.

(2) The distance of the centre of the circle from BC is $\tfrac{1}{2}a\cot A$; let $lp + mq + nr = 0$ be the equation of the centre; and the values of p, q, r for a plane through BC, perpendicular to plane ABC, are

$b\sin C$, 0, 0, and, by Art. 116, $lb\sin C/(l+m+n)=\frac{1}{2}a\cot A$; $\therefore l/(l+m+n)=\frac{1}{2}\cos A/\sin B\sin C$, $\therefore l/\sin 2A=m/\sin 2B=n/\sin 2C$.

Aliter. Let O be the centre and let AO produced cut BC in a, shew that $Ba : aC=\sin 2C : \sin 2B$, equation of a is

$$q\sin 2B+r\sin 2C=0.$$

(3) The plane passes through the three points $p+q+r=0$, $p+r+s=0$, and $p+s+q=0$, $\therefore q=r=s=-\frac{1}{2}p$, and each is equal to $\frac{1}{3}p_0$, either directly by geometry, or by Art. 127.

(4) Let $DP:PB=\lambda:\mu$; then the equations of P and Q are $\lambda q+\mu s=0$ and $\mu p+\lambda r=0$, $\lambda(q+r)+\mu(p+s)=0$ is that of a point in PQ, which lies in the line joining the middle points of BC, AD, dividing it in ratio $\mu:\lambda$.

(5) $q+r=0$, $p-s=0$ are the equations of the middle point of BC, and of the point at infinity in AD, Art. 121. $r+s=0$, $p-q=0$ are similar equations for CD and AB, $q+s=0$, $p-r=0$ for DB, AC, hence the three lines of the problem have a common point, whose equation is $q+r+s-p=0$.

(6) Let $\lambda p+\mu q+\nu r+\rho s=0$ be the equation required, the distance from BCD is

$$\lambda p_0/(\lambda+\mu+\nu+\rho)=\tfrac{1}{3}p_0(B+C+D)/(A+B+C+D),$$
$$\therefore \lambda:\mu:\nu:\rho=B+C+D : C+D+A : D+A+B : A+B+C.$$

(7) The tetrahedral coordinates of the centre of any sphere touching the planes of the four faces are proportional to A, B, C, D with the proper signs, and the equations of the centres of such spheres must be included in the form $\pm Ap\pm Bq\pm Cr\pm Ds=0$; four such as $Ap+Bq+Cr-Ds=0$, three such as $Ap+Bq=Cr+Ds$, and one $Ap+Bq+Cr+Ds=0$.

XII.

(1) Let G be the centre of gravity of ACD, the equations of G and B are $\frac{1}{3}(p+r+s)=0$, $q=0$; $\therefore$ that of b is

$$\tfrac{2}{3}(p+r+s)-q=0, \text{ or } 2(p+q+r+s)=5q;$$

$\therefore q=r=s=-2p$ for the plane bcd, cutting AB at a point $2p+q=0$.

(2) The point in the line joining B to the centre of gravity of ACD is $m.\frac{1}{3}(p+r+s)+nq=0$, or $m(p+q+r+s)=(m-3n)q$; and, for the proposed plane, $q=r=s$, and $mp+(2m+3n)q=0$.

(3) x, y, z, w being tetrahedral coordinates of P; for a, $yq+zr+ws=0$; for b, $xp+zr+ws=0$; where ab meets AB, $xp-yq=0$; for d, $xp+yq+zr=0$; where Cd meets AB, $xp+yq=0$.

(4) P, Q being intersections of AB, ab and DB, db; for P and D, $xp-yq=0$, and $s=0$; for Q and A, $yq-ws=0$, and $p=0$; $\therefore xp+ws-yq=0$ gives the point of intersection of PD and QA, and since it is $xp+yq+ws-2qy=0$, the theorem is true.

(5) The equation of H is
$(B+C+D)(p+q+r+s)-Ap-Bq-Cr-Ds=0$, see XI. (6), in the perpendicular form of Art. 118, the equations of O and G are $(Ap+Bq+Cr+Ds)/(A+B+C+D)=0$ and $\frac{1}{4}(p+q+r+s)=0$;

$$\therefore HO : HG :: 4 : 1, \text{ or } OG = 3GH.$$

(6) Let (p', q', r', s') (p'', q'', r'', s'') be the two planes U', U''; and let λ', μ', ν', ρ' be the cosines of the angles between the normal to the plane U', and the normals to the faces of the fundamental tetrahedron. Let a perpendicular from A on the plane U' meet U'' in N, then $AN=\varpi=p''\sec(U', U'')$. Tetrahedral coordinates of N are $(p_0-\varpi\lambda')/p_0$, $-\varpi\mu'/q_0$, $-\varpi\nu'/r_0$, $-\varpi\rho'/s_0$, $\therefore$ the equation of N is $(p_0-\varpi\lambda')p/p_0-\varpi\mu' q/q_0-\varpi\nu' r/r_0-\varpi\rho' s/s_0$, and U'' is a particular plane through N;

$$\therefore \cos(U', U'')=p''/p_0=\lambda' p''/p_0+\mu' q''/q_0+\nu' r''/r_0+\rho' s''/s_0.$$

By Art. 126, $\lambda'=p'/p_0-\cos(AB)q'/q_0-\cos(AC)r'/r_0-\cos(AD)s'/s_0$, and similarly for μ', ν', ρ', the given result follows.

XIII.

(1) (i) Tetrahedral coordinates. $\alpha=0$, $\gamma=0$ is a solution, therefore every point in BD is on the surface; similarly AC, BC and AD lie entirely in the surface.

Four-point coordinates. From the solution $\alpha=0$, $\gamma=0$ any plane through AC touches the locus; let $\lambda\alpha+\nu\gamma=0$ be a point P in AC, by the given equation, $\gamma=0$ and $l\nu\beta+m\lambda\delta=0$, give two points C, and Q in BD, such that planes through PC and PQ are tangents to the locus, $\therefore$ CPQ is a tangent plane to the locus, and P the point of contact. It follows that AC, AD, BC, BD lie entirely in the locus.

(ii) Tetrahedral coordinates. When $\gamma=0$, $(\alpha+\beta)^2=0$ (1), therefore plane ABD touches the surface at every point where it meets a plane through CD parallel to AB. A plane $\lambda(\alpha+\beta)=\mu\gamma$ intersects the surface in another plane $\mu(\alpha+\beta)=\lambda n\delta$ (2), the surface is generated by lines parallel to AB, guided by a conic in ACD touching AC, AD at C and D.

Four-point coordinates. By the equations (1) any plane through C and the middle point Q of AB touches the locus at C; and equations (2) determine points R, S in QC and QD, such that any plane through RS is a tangent plane to the locus; the locus is therefore a curve in the plane QCD, touching QC and QD at C and D.

(iii) Write the equation $l\beta\gamma\delta + m\gamma\delta\alpha + n\delta\alpha\beta + r\alpha\beta\gamma = 0$.

Tetrahedral coordinates. $\alpha = 0$, $\beta = 0$ (3) satisfy the equation, $\therefore$ CD lies entirely in the surface; so for all the edges.

$$\alpha/l + \beta/m = 0 \text{ and } \gamma/n + \delta/r = 0 \quad (4)$$

satisfy the equation, and the line of intersection of the two planes through the opposite edges CD and AB lies entirely in the surface; the three corresponding lines for the opposite edges all lie in one plane $\alpha/l + \beta/m + \gamma/n + \delta/r = 0$ (5).

Four-point coordinates. Equations (3) shew that every plane through AB touches the locus, similarly for all the edges. Equations (4), joining points P, Q in AB and CD, shew that all planes containing P and Q touch the locus, and that the three corresponding lines joining pairs of opposite edges all pass through a point, whose equation is (5).

(2) The surface represented by $u = 0$ is the envelope U of all the planes whose coordinates satisfy the equation, $v = 0$ is the equation of a point V. Let $v' = 0$ be the equation of a point V' near V, and let C, C' be the curves on U, which are the loci of the points of contact of tangent planes through V and V'; the surface represented by $vv' = Au$ touches both the cones, whose vertices are V, V', and whose generating lines are guided by the curves C and C'.

When V' moves up to V, C and C' coincide, and the surface represented by $v^2 = Au$ touches the coincident cones along the curve C, and therefore touches U along the same curve.

(3) The equation of the centre is $\frac{1}{4}(p+q+r+s) = 0$ in both cases; the radii of the two spheres are $\frac{3}{4}p_0$ and $\frac{1}{4}p_0$. By Art. 119, the distance of the centre from the tangent plane (p, q, r, s) is $\frac{1}{4}(p+q+r+s)$, which $= \frac{3}{4}p_0$ or $\frac{1}{4}p_0$; and, by Art. 127,

$$p^2 + q^2 + r^2 + s^2 - \tfrac{2}{3}(pq + qr + \ldots) = p_0^2.$$

(4) Compare the given equation with that of a sphere of radius ρ, viz. $\{\frac{1}{4}(p+q+r+s)\}^2 p_0^2 = \rho^2\{p^2 + \ldots - \frac{2}{3}(pq + \ldots)\}$; and shew that $\rho^2 = \frac{3}{16}p_0^2$, and that ρ is half the distance of the opposite edges.

(5) The distances required are $\rho \pm \frac{1}{4}p_0$, where $p_0 = a\sqrt{\frac{2}{3}}$.

(6) The torse consists of two cones, the vertices corresponding to the planes of the two curves.

(7) Let $\alpha x + \beta y + \gamma z = 1$ be the Cartesian equation of a tangent plane, r is the perpendicular distance of the centre (lr, mr, nr) from the tangent plane, $\therefore \{1 - (l\alpha + m\beta + n\gamma)r\}/\sqrt{(\alpha^2 + \beta^2 + \gamma^2)} = r$.

(8) Let α, β, γ be Boothian coordinates of a common tangent plane, $\therefore r^{-1} - \alpha = \sqrt{(\alpha^2 + \beta^2 + \gamma^2)} = s^{-1} - \beta$.

XIV.

(1) Since $am_1n_1+bn_1l_1+cl_1m_1=0$ and $am_2n_2+bn_2l_2+cl_2m_2=0$, (1) also $m_1n_1+m_2n_2+m_3n_3=0$, &c., $\therefore am_3n_3+...=0$.

Eliminating c from (1),

$$a\,(l_2m_2m_1n_1-l_1m_1m_2n_2)=b\,(l_1m_1m_2l_2-l_2m_2m_1l_1),$$

and, by Art. 146,

$$l_2n_1-l_1n_2 : n_2m_1-n_1m_2=m_3 : l_3, \ \therefore am_1m_2m_3=bl_1l_2l_3.$$

(2) By addition $a+b+c=0$, $\therefore a\,(l_1^2-n_1^2)+b\,(m_1^2-n_1^2)=0$, and $a:b:c=m_1^2-n_1^2 : n_1^2-l_1^2 : l_1^2-m_1^2$, &c.

Also $a\,(l_2^2n_3^2-l_3^2n_2^2)=b\,(n_2^2m_3^2-n_3^2m_2^2)$,

hence, by Art. 146,

$$a:b:c=l_1\,(m_2n_3+m_3n_2) : m_1\,(n_2l_3+n_3l_2) : n_1\,(l_2m_3+l_3m_2).$$

(3) Turn the axes Ox, Oy through $45°$, $x=\sqrt{\tfrac{1}{2}}\,(x'-y')$ and $y=\sqrt{\tfrac{1}{2}}\,(x'+y')$; $\therefore \sqrt{2}x'z+\tfrac{1}{2}(x'^2-y'^2)=a^2$, turn Ox', Oz through angle $\cos^{-1}\sqrt{\tfrac{2}{3}}$, $x'=\sqrt{\tfrac{2}{3}}x''-\sqrt{\tfrac{1}{3}}z''$, $z=\sqrt{\tfrac{1}{3}}x''+\sqrt{\tfrac{2}{3}}z''=0$, $x''^2-\tfrac{1}{2}(y''^2+z''^2)=a^2$.

(4) As in (3) the equation becomes $x^2+y^2+z^2+x^2-\tfrac{1}{2}(y^2+z^2)=a^2$.

(5) Let $O\lambda$, $O\mu$, $O\nu$ be each perpendicular to two of Ol, Om, On, viz. $O\lambda$ to Om and On, &c., and also perpendicular to each other; Ol is perpendicular to $O\mu$ and $O\nu$ and the plane $\mu O\nu$.

Analytically, $\lambda_2l_1+\mu_2m_1+\nu_2n_1=0$ and $\lambda_3l_1+\mu_3m_1+\nu_3n_1=0$, $\therefore l_1 : m_1 : n_1=\mu_2\nu_3-\mu_3\nu_2 : ... : ...=\lambda_1 : \mu_1 : \nu_1$.

(6) As in Art. 148, $a\alpha\alpha'+b\,(\alpha\beta'+\alpha'\beta)+c\beta\beta'=0$ and $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0$. The required equation is to be of the form $Ax^2+Bxy+Cy^2=0$, which must reduce by transformation to $x'y'=0$,

$$\therefore A\alpha^2+B\alpha\beta+C\beta^2=0, \text{ and } A\alpha'^2+B\alpha'\beta'+C\beta'^2=0;$$

$$\therefore A : B : C=-\beta\beta' : \alpha\beta'+\alpha'\beta : -\alpha\alpha'.$$

Now, since the bisectors are in the plane $lx+my+nz=0$,

$$l\alpha+m\beta+n\gamma=0, \quad l\alpha'+m\beta'+n\gamma'=0,$$

$$\therefore l^2\alpha\alpha'+lm\,(\alpha\beta'+\alpha'\beta)+m^2\beta\beta'=n^2\gamma\gamma'=-n^2\,(\alpha\alpha'+\beta\beta');$$

$$\therefore (m^2+n^2)\,A-lmB+(l^2+n^2)\,C=0, \text{ and } cA-bB+aC=0;$$

$$\therefore A : B : C=alm-b(l^2+n^2) : a(m^2+n^2)-c\,(l^2+n^2) : -clm+b(m^2+n^2).$$

(7) The required equation being $Ax^2+By^2+Cz^2=0$, this must be transformed to $x'y'=0$,

$$\therefore A\alpha^2+B\beta^2+C\gamma^2=0, \text{ and } A\alpha'^2+B\beta'^2+C\gamma'^2=0,$$

whence $A\,(\alpha^2\gamma'^2-\alpha'^2\gamma^2)=B\,(\gamma^2\beta'^2-\gamma'^2\beta^2)$, whence, by Art. 146,

$$Am\,(\alpha\gamma'+\alpha'\gamma)=Bl\,(\gamma\beta'+\gamma'\beta),$$

as in prob. (6), $l^2\alpha\alpha'+m^2\beta\beta'-n^2\gamma\gamma'=-lm\,(\alpha\beta'+\alpha'\beta)$, &c.,

and $a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'=0$, $\alpha\alpha'+\beta\beta'+\gamma\gamma'=0$;

$$\therefore \alpha\alpha' : \beta\beta' : \gamma\gamma'=b-c : c-a : a-b;$$

$$\therefore Am^2\{l^2(b-c)-m^2(c-a)+n^2(a-b)\}=Bl^2\{-l^2(b-c)+m^2(c-a)+n^2(a-b)\}.$$

(8) Equating the coefficient of $x'y'$ to zero, as in Art. 148,

$$a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' + a'(\beta\gamma' + \beta'\gamma) + b'(\gamma\alpha' + \gamma'\alpha) + c'(\alpha\beta' + \alpha'\beta) = 0,$$

$$\alpha' : \beta' : \gamma' = m\gamma - n\beta : n\alpha - l\gamma : l\beta - m\alpha,$$

substitute for α', β', γ', and the equation resulting is

$$(c'n - b'm)\alpha^2 + \ldots + \{c'm - b'n + (c - b)l\}\beta\gamma + \ldots = 0.$$

Similarly

$$(c'n - b'm)\alpha'^2 + \ldots + \{c'm - b'n + (c - b)l\}\beta'\gamma' + \ldots = 0;$$

hence the bisectors lie in the given cone.

(9) Let λ, μ, ν and λ', μ', ν' be the inclinations of the axes in the two cases, and if P be written for

$$1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu,$$

the last term in the cubics of Art. 157, namely

$$h^3 + Ah^2 + Bh - abc/P = 0, \text{ and } h^3 + A'h^2 + B'h - \alpha\beta\gamma/P' = 0,$$

will be equal, and, since P and P' are positive, abc and $\alpha\beta\gamma$ will have the same sign.

(10) After transformation, let $x^2 + y^2 + \frac{1}{2}yz + zx \equiv u$ become $\alpha x^2 + \beta y^2 + \gamma z^2$, the values of h, which make $h(x^2 + y^2 + z^2) - u$ the product of two linear factors, are the roots of

$$(h-1)^2 h - \tfrac{1}{16}(h-1) - \tfrac{1}{4}(h-1) = 0,$$

viz. 1, $\frac{5}{4}$, and $-\frac{1}{4}$; on transformation $h(x^2 + y^2 + z^2) - \alpha x^2 - \beta y^2 - \gamma z^2$ is still the product of two factors with the same values of h; $\therefore\ \alpha = 1,\ \beta = \frac{5}{4},\ \gamma = -\frac{1}{4}$.

(11) Let A', B', C', D' be the centres of gravity of the faces opposite to A, B, C, D; the equation of A' is $\frac{1}{3}(q + r + s) = 0$, and for any plane, as in Art. 160, $p' = \frac{1}{3}(q + r + s)$, $r' =$&c., the equation of the centre of gravity of $A'B'C'D'$ is $\frac{1}{4}(p' + q' + r' + s') = \frac{1}{4}(p + q + r + s)$, *i.e.* the centres of gravity coincide.

(12) The equation of the centre of the sphere is

$$Ap' + Bq' + Cr' + Ds' = 0, \text{ see prob. XI. (7)},$$

or $\frac{1}{3}A(q + r + s) + \ldots = 0$ referred to $ABCD$, which is the equation of the centre of gravity of the surface of $ABCD$, see prob. XI. (6).

XV.

(1) Take O the origin, and the fixed plane parallel to that of xy cutting Oz in C, let $OC = c$, and let ac be the given quantity,

$$OQ : OP = c : z = ac : OP^2.$$

(2) Take the plane of xy that of the given circle, the origin at the centre, and the plane of yz parallel to that of the variable circle, $x = a\cos\theta$, $y^2 + z^2 = a^2\sin^2\theta$ its equations.

(3) Use figure, page 3. Draw MA perpendicular to OM meeting Ox in A, locus of M is a circle diameter $OA=a$, locus of P in plane POM is a circle diameter OM.

(4) l, m, n direction-cosines of AP; $\therefore x=l.AP$, &c.

(5) Take the given point for origin.

i. Let plane of yz be parallel to the given plane, $x=a$ its equation; that of locus will be $x^2+y^2+z^2=e^2(x-a)^2$, $e<1$ a prolate spheroid, $e=1$ an elliptic paraboloid, $e>1$ a hyperboloid of revolution of two sheets.

ii. Let plane of zx contain the given line, and Ox be parallel to it at a distance a. The equation of the locus will be

$$x^2+y^2+z^2=e^2\{y^2+(z-a)^2\},$$

$e<1$ an oblate spheroid, $e=1$ a parabolic cylinder, $e>1$ a hyperboloid of revolution of one sheet.

(6) Axes as in Art. 64. By the method of Art. 58, the equation is $x^2+y^2+(z-c)^2-(x\cos\alpha+y\sin\alpha)^2=x^2+y^2+(z+c)^2-(x\cos\alpha-y\sin\alpha)^2$, or $xy\sin\alpha\cos\alpha+cz=0$ a hyperbolic paraboloid.

(7) Take Ox for the fixed line, the origin being where the line and plane intersect, and let the plane xy contain the line and its projection on the plane, $x\sin\alpha-y\cos\alpha=0$ the equation of the plane; that of the locus will be $y^2+z^2=(x\sin\alpha-y\cos\alpha)^2$.

(8) Let the planes yx and zx contain the parabolas, latera recta $2l$ and $2l'$, the equation of the ellipse, the distance of whose plane from yz is x, is $y^2/lx+z^2/l'x=1$.

(9) Let λa, λb, λc be semi-axes of a degenerated hyperboloid, its equation is $x^2/a^2\pm y^2/b^2-z^2/c^2=\lambda^2$.

(10) Let $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$ be the direction-cosines of the lines; $l_1^2/a^2+m_1^2/b^2+n_1^2/c^2=r_1^{-2}$, &c., and $l_1^2+l_2^2+l_3^2=1$, &c.; $\therefore a^{-2}+\ldots=r_1^{-2}+\ldots$.

(11) The form of the section by a plane, whose inclination to that of xy is θ, is given by $y=r\cos\theta$, $z=r\sin\theta$, whence $r^2=4\alpha x$, where $\frac{1}{4}\alpha^{-1}=\cos^2\theta/b+\sin^2\theta/c$; for the extremity of the latus rectum, $x=\alpha$, $y=2\alpha\cos\theta$, $z=2\alpha\sin\theta$.

(12) Axes as in Art. 64. Planes through the two lines have equations $x\sin\alpha-y\cos\alpha+\lambda(z/c-1)=0$ and $x\sin\alpha+y\cos\alpha+\lambda'(z/c+1)=0$, and the planes are at right angles, $\therefore \sin^2\alpha-\cos^2\alpha+\lambda\lambda'/c^2=0$;

$$\therefore x^2\sin^2\alpha-y^2\cos^2\alpha=\lambda\lambda'(z^2/c^2-1)=\cos 2\alpha\,(z^2-c^2).$$

XVI.

(1) Equation of any sphere containing the circle is

$$(x-a)^2+(y-b)^2-r^2+z^2+2Az=0,$$

when $y=0$ the section is an indefinitely small circle,

$$\therefore\ (x-a)^2+(z+A)^2=0, \quad \therefore\ A^2=b^2-r^2;$$

$$\text{when } x=0,\ (y-b)^2+(z+A)^2=A^2+r^2-a^2=b^2-a^2.$$

The magnitude of the circle in yz depends only on the radius of the sphere and the distance of its centre from yz, which, from the construction, are b and a respectively.

(2) Let Oz be the fixed line, and let $OQ=a$, the shortest distance between the two lines, be in the plane xy, α the angle between the lines, (x, y, z) a point P in the revolving line, PM perpendicular to plane xy, $OM^2=OQ^2+QM^2$; $\therefore\ x^2+y^2=a^2+z^2\tan^2\alpha$.

(3) Let a plane through the fixed point A, perpendicular to the line of intersection Oz of the given planes, cut the two planes in Ox, Oy. The middle point of any of the lines cut off by the two planes projects orthogonally into that of the line through A cut off by Ox, Oy. Let (ξ, η) be the middle point of the projection of the line in the plane xy, (a, b) the point A. Its equation is $x/2\xi+y/2\eta=1$, and it passes through A, $\therefore\ a/\xi+b/\eta=2$.

(4) Take Oz the line to which the moving line is perpendicular, Ox perpendicular to Oz and the other given line, whose equations are $x=a$, $z=my$, those of the moving line $z=\alpha$, $y=\beta x$; $\therefore$ since they intersect $\alpha=m\beta a$ and $xz=may$. When $m=0$, $z=0$, or $x=0$, every line in plane xy through O satisfies the conditions, and every line parallel to Oy through Oz meets the other line at infinity.

(5) The given equations may be written $b(x^2+y^2)+(a-b)m^2z^2=0$, and $x=mz$, and when the ellipse rotates, neither z nor x^2+y^2 are altered.

(6) As in Art. 175, the equation of the cone is

$$(hx-fz)^2/a^2+h^2y^2/b^2=(z-h)^2,$$

where $f^2/(a^2-b^2)=1+h^2/b^2$; when it is circular a sphere, centre in the vertex, cuts it in two parallel planes. Changing the origin to the vertex, the equation, by the given relation between f and h, reduces to $(x^2+y^2+z^2)\,a^2h^2\,(a^2-b^2)-\{(a^2-b^2)\,hx-b^2fz\}^2=0$.

(7) Let the plane to which the circles are parallel pass through a diameter of the fixed circle, take this diameter for Oy, the origin in the centre, the plane of xy bisecting the angle between the two planes; the centre of the moving circle will move along either the axis of x or z, and generate two corresponding cylinders.

(8) The equations of the two cones are $y^2/b^2+z^2/c^2=(x-a)^2/a^2$, and $x^2/a^2+z^2/c^2=(y-b)^2/b^2$; the cones intersect where

$$x/a=y/b=r/\sqrt{(a^2+b^2)}, \text{ and } z^2/c^2=1-2r/\sqrt{(a^2+b^2)};$$
$$\therefore\ l^{-1}=\sqrt{(a^2+b^2)}/2c^2.$$

(9) Let the equation of the plane be $(x-ae)/\cos\theta=y/\sin\theta=r$, where it meets the given surface,

$$(ae+r\cos\theta)^2/(a^2-\lambda^2)+(r^2\sin^2\theta+z^2)/(c^2-\lambda^2)=1,$$

the condition makes the coefficient of r^2 vanish, and the latus rectum $=2ae\cos\theta\,(\lambda^2-c^2)/(a^2-\lambda^2)$, since $a^2>\lambda^2>c^2$.

(10) Take the axes as in Art. 64, the first line being in the moving plane, $y\cos\alpha-x\sin\alpha+A(z-c)=0$, a plane through the other line is $y\cos\alpha+x\sin\alpha+B(z+c)=0$; if these planes be perpendicular $\cos^2\alpha-\sin^2\alpha+AB=0$, and their intersection is the projection of the given line, $\therefore\ y^2\cos^2\alpha-x^2\sin^2\alpha=(\sin^2\alpha-\cos^2\alpha)(z^2-c^2)$, a hyperboloid of one sheet, including circular cylinders when $\alpha=0$ or $\frac{1}{2}\pi$. When the two lines are at right angles, $B=0$, and $x+y=0$ is a plane perpendicular to the moving plane in all positions.

(11) If FG, PQ be perpendiculars from $F(f,g,h)$ and P on the line $x/A=$&c., OP and GQ are constant while FP revolves aboot OGQ. For the original position of FP, coordinates are $f+lp$ &c., $\therefore\ OP^2=x^2+y^2+z^2=(f+lp)^2+\ldots$ and

$$A(x-f)+\ldots=GQ\sqrt{(A^2+B^2+C^2)}=Alp+Bmp+Cnp.$$

(12) Let (ξ,η,ζ) be the point, r_1, r_2, r_3 distances from the conic, $l_1m_1n_1$, $l_2m_2n_2$, $l_3m_3n_3$, their direction-cosines.

$$a(\xi-l_1r_1)^2+b(\eta-m_1r_1)^2=1,\quad \zeta=n_1r_1,$$
$$\therefore\ (a\xi^2+b\eta^2-1)\,n_1^{\,2}-2a\zeta\xi l_1n_1-2b\zeta\eta m_1n_1+(al_1^{\,2}+bm_1^{\,2})\,\zeta^2=0.$$

Similarly for r_2 and r_3, $\therefore\ a\xi^2+b\eta^2-(a+b)\,\zeta^2=1$.

(13) As in Art. 175, the equation of the cone is

$$(xh-fz)^2/a^2+(hy-gz)^2/b^2=(z-h)^2,$$
$$\text{or}\quad h^2(x^2/a^2+y^2/b^2+z^2/c^2-1)+(f^2/a^2+g^2/b^2-h^2/c^2)\,z^2$$
$$-2zh\,(fx/a^2+gy/b^2-1)=0,$$

the two planes of intersection are $z=0$ and

$$2h\,(fx/a^2+gy/b^2-1)-(f^2/a^2+g^2/b^2-h^2/c^2)\,z=0,$$

in the latter if $z=0$, $fx/a^2+gy/b^2=1$.

(14) Let 2α be the vertical angle of the cone, the equation of the plane through the origin perpendicular to the axis $\lambda x+\mu y+\nu z=0$; then $\cos\alpha=l_1\lambda+m_1\mu+n_1\nu=l_2\lambda+m_2\mu+n_2\nu=l_3\lambda+m_3\mu+n_2\nu$.

XVII.

(1) See figure p. 92. If the figure represent a hyperboloid of revolution, and PTP' be a constant angle, the locus of T will be a circle; Q, Q' lie on two circles whose planes are parallel to the locus of T. Let every line measured in a direction perpendicular to the principal plane $RAA'R'$ be diminished in a constant ratio, the eccentric angles of P and P' differ by a constant angle and the three planes remain parallel.

(2) By Art. 209 a hyperboloid can be constructed of which three lines (A), (B), (C) are generators of the same system, a fourth line (D) meets the hyperbloid in only two points P, Q, and two generators through P and Q of the system opposite to that of (A), (B), (C) intersect all four lines.

(3) Let α be the eccentric angle of the point on the principal elliptic section through which a generator passes, whose equations are $(x - a\cos\alpha)/a\sin\alpha = (y - b\sin\alpha)/(-b\cos\alpha) = \pm z/c$, Art. 213; if this line meet the sections by the planes of zx, zy in the points at which ϕ, ϕ' are the eccentric angles,

$$\sec\phi - \cos\alpha = \sin\alpha\tan\alpha = \pm\sin\alpha\tan\phi,$$
$$\cot\alpha\cos\alpha = \sec\phi' - \sin\alpha = \mp\cos\alpha\tan\phi',$$
$$\therefore\ \tan\phi = \pm\tan\alpha = -\cot\phi'.$$

(4) Generators corresponding to an eccentric angle α will be at right angles, if $a^2\sin^2\alpha + b^2\cos^2\alpha = c^2$.

(5) By the method of Art. 210, (l, m, n) being the direction of a generator through (ξ, η, ζ) of the paraboloid $y^2/b^2 - z^2/c^2 = 2x/a$, $m^2/b^2 = n^2/c^2$, and $m\eta/b^2 - n\zeta/c^2 = l/a$,

$$\text{whence } l : m : n = a(\eta/b \mp \zeta/c) : b : \pm c,$$
$$\therefore\ b^2 - c^2 + a^2(\eta^2/b^2 - \zeta^2/c^2) = 0 \text{ or } 2a\xi + b^2 - c^2 = 0.$$

(6) Take the plane yz parallel to two of the lines and containing the third, and planes zx, xy containing the first two. Let the equations be $x = a$, $z = 0$; $x = b$, $y = 0$; $x = 0$, $z = my + c$; and those of the generating line $z = \alpha(x - a)$, $y = \beta(x - b)$; $\therefore\ \alpha a - m\beta b + c = 0$; showing that the generator is parallel to the plane $az - mby + cx = 0$;

(7) As in Art. 210, if (λ, μ, ν) be the direction of a generator, $\mu\nu + \nu\lambda + \lambda\mu = 0$ and $\nu m - \mu/m + \lambda(m - m^{-1}) = 0$,

$$\text{whence } \lambda(1 \pm m) = -\mu = \pm\nu m.$$

(8) For a generator through (ξ, η, ζ) in direction (λ, μ, ν)

$$\mu\nu + \nu\lambda + \lambda\mu = 0 \text{ and } (\eta + \zeta)\lambda + (\zeta + \xi)\mu + (\xi + \eta)\nu = 0;$$

hence, if $\lambda_1\lambda_2$ &c., be the values of λ for the two generators,

$$\lambda_1\lambda_2(\eta + \zeta) = \mu_1\mu_2(\zeta + \xi) = \nu_1\nu_2(\xi + \eta)$$
$$\text{and } (\xi + \eta)(\xi + \zeta) + (\eta + \zeta)(\eta + \xi) + (\zeta + \xi)(\zeta + \eta)$$
$$= (\xi + \eta + \zeta)^2 + \eta\zeta + \zeta\xi + \xi\eta = 0.$$

(9) Take the three generators for coordinate axes, the cone's axis is equally inclined to the axes, and if 2α be the vertical angle, $\cos\alpha = \sqrt{\frac{1}{3}}$.

(10) By Art. 214, $\theta_1+\phi_1=\theta_2+\phi_2$, $\theta_1-\phi_1=\theta_4-\phi_4$, $\theta_3+\phi_3=\theta_4+\phi_4$, and $\theta_3-\phi_3=\theta_2-\phi_2$ which give the result.

(11) Let the equations of the generator be $y/\sqrt{a} \pm z/\sqrt{b} = \alpha$ and $x = \gamma y + \delta$, since it intersects the two parabolas, $\alpha^2 - \gamma\alpha\sqrt{a} - \delta = 0$ and $\alpha^2 + \delta = 0$; $\therefore\ 2\alpha^2 - \gamma\alpha\sqrt{a} = 0$, and $x = y^2/a - (\alpha - y/\sqrt{a})^2$.

(12) For two generators which intersect,

$$x/a = \cos(\alpha+\beta) + \sin(\alpha+\beta)z/c \text{ and } x/a = \cos(\alpha-\beta) - \sin(\alpha-\beta)z/c,$$

$$\text{whence } x/a = \cos\alpha\cos\beta + \cos\alpha\sin\beta\, z/c,$$

$$\text{and } 0 = \sin\alpha\sin\beta - \sin\alpha\cos\beta\, z/c,$$

$$\therefore\ \cos\beta\, x/a = \cos\alpha, \text{ similarly } \cos\beta\, y/b = \sin\alpha.$$

Again, for two generators which do not intersect,

$$\{x/a - \cos(\alpha\pm\beta)\}/\sin(\alpha\pm\beta) = \{y/b - \sin(\alpha\pm\beta)\}/-\cos(\alpha\pm\beta) = z/c,$$

$$\text{let } A\{x/a - \cos(\alpha+\beta)\} + B\{y/b - \sin(\alpha+\beta)\} + Cz/c = 0$$

be the equation of a plane containing one of the generators and parallel to the others, $\therefore\ A\sin(\alpha\pm\beta) - B\cos(\alpha\pm\beta) + C = 0$,

$$\text{whence } A/\sin\alpha = B/-\cos\alpha = C/-\cos\beta;$$

$$\text{and } \sin\alpha\, x/a - \cos\alpha\, y/b - \cos\beta\, z/c + \sin\beta = 0,$$

$$\text{and } \sin\alpha\, x/a - \cos\alpha\, y/b - \cos\beta\, z/c - \sin\beta = 0$$

are the equations of two planes each containing one generator and parallel to the other, and δ is the difference of the perpendiculars from the origin.

XVIII.

(1) The shortest distance between two, being perpendicular to both, is parallel to the third, and therefore meets it at an infinite distance, hence it is a generating line.

(2) The equations of the two planes are given by

$$A(y/b - z/c) + B(1 - x/a) = 0 \quad (1),$$

$$\text{and } B(y/b + z/c) + A(1 + x/a) = 0 \quad (2),$$

the planes on which the traces are made must be parallel to the axis of x, for the positions $A = 0$, $B = 0$; let $y = mz$ be a plane on which the traces are made, the direction-cosines of its intersection with (1) and (2) are respectively as

$$AB^{-1}a(mb^{-1} - c^{-1}) : m : 1, \text{ and } BA^{-1}a(mb^{-1} + c^{-1}) : m : 1;$$

$\therefore\ a^2(m^2b^{-2} - c^{-2}) + m^2 + 1 = 0$ gives the two positions of the fixed planes, independent of $A : B$.

(3) Consider any hyperboloid of revolution $x^2+y^2-m^2z^2=a^2$, the tangent plane to the section by the plane of yz determines two generators inclined at equal angles to the axis of x, if therefore a ray of light coinciding with one generator be reflected at the plane yz, the reflected ray will coincide with the other generator; the same will be true for reflection at any plane passing through Oz. If therefore two mirrors intersect in OZ, a ray will after every reflection be a generator of a hyperboloid of revolution, whose axis is the intersection of the mirrors, and of which the incident ray is a generator.

(4) Since (l, m, n) is the direction of the line represented by any two of the equations, and we can deduce from them the equations $l^2a+m^2b+n^2c=0$ and $lax+mby+ncz=0$, therefore the conditions, Art. 210, of being a generator through (x, y, z) are satisfied.

If l, m, n and l_1, m_1, n_1 be the two solutions of the equations

$$a\lambda^2+b\mu^2+c\nu^2=0, \text{ and } af\lambda+bg\mu+ch\nu=0,$$

shew that $ll_1 : mm_1 : nn_1 = bc+f^2 : ca+g^2 : ab+h^2$, hence that the other generator through any point (f, g, h) in the first is

$$l(x-f)/(bc+f^2)=m(y-g)/(ca+g^2)=n(z-h)/(ab+h^2).$$

(5) For the generators $x/a \pm y/b = 2z/\lambda$ (1), and $x/a \mp y/b = \lambda/c$, the direction-cosines are as $a : \pm b : \lambda$; let (l, m, n) be the direction of the perpendicular from the origin, $\therefore la \pm mb + n\lambda = 0$, and since it is in the plane (1), $l/a \pm m/b = 2n/\lambda$, $\therefore (l/a \pm m/b)(la \pm mb)+2n^2=0$.

(6) For a generator let

$$x/a - z/c = \lambda(1-y/b) \text{ and } x/a+z/c=\lambda^{-1}(1+y/b),$$

then the plane containing the origin and generator is

$$(\lambda-\lambda^{-1})\,x/a - 2y/b + (\lambda+\lambda^{-1})\,z/c = 0,$$

and the direction-cosines of the generator are as

$$-a(\lambda-\lambda^{-1}) : 2b : c(\lambda+\lambda^{-1}).$$

Let (l, m, n) be the direction of the perpendicular,

$$(\lambda-\lambda^{-1})\,l/a - 2m/b + (\lambda+\lambda^{-1})\,n/c = 0,$$

$$\text{and } -(\lambda-\lambda^{-1})\,al + 2mb + (\lambda+\lambda^{-1})\,nc = 0.$$

Find $\lambda+\lambda^{-1}$ and $\lambda-\lambda^{-1}$ and take the difference of their squares.

(7) Write the equation $ax^2+by^2+cz^2=1$, and let $\lambda x+\mu y+\nu z=0$ be the equation of one of the planes containing a generator through (f, g, h) and intersecting the hyperboloid in points lying in the plane $\alpha x+\beta y=1$. For an infinite number of values x and y

$$\nu^2(ax^2+by^2)+c(\lambda x+\mu y)^2=\nu^2(\alpha x+\beta y)^2,$$

hence, equating to zero the coefficients of x^2, xy, y^2, and eliminating α and β, $\lambda^2a^{-1}+\mu^2b^{-1}+\nu^2c^{-1}=0$, also $\lambda f+\mu g+\nu h=0$,

$\therefore\ h^2(\lambda^2 a^{-1}+\mu^2 b^{-1})+c^{-1}(\lambda f+\mu g)^2=0$, whence for the two directions

$$\lambda_1\lambda_2 : \mu_1\mu_2 : \lambda_1\mu_2+\lambda_2\mu_1 : \lambda_1\mu_2-\lambda_2\mu_1$$
$$=a(1-af^2) : b(1-bg^2) : -2abfg : \sqrt{(-4abch^2)}$$
$$\lambda_1\lambda_2+\ldots : \sqrt{\{(\lambda_1\mu_2-\lambda_2\mu_1)^2+\ldots\}}=a+b+c-p^{-2} : 2r\sqrt{(-abc)}.$$

(8) Let (l, m, n) be the direction of the perpendicular on a generator, $\therefore\ la\sin\alpha-mb\cos\alpha\pm nc=0$, and

$$(lr-a\cos\alpha)/a\sin\alpha=(mr-b\sin\alpha)/(-b\cos\alpha)=nr/\pm c$$
$$=(a^2-b^2)\sin\alpha\cos\alpha/(a^2\sin^2\alpha+b^2\cos^2\alpha+c^2),$$

hence prove that

$$l : m : n=a(b^2+c^2)\cos\alpha : b(a^2+c^2)\sin\alpha : \pm c(a^2-b^2)\sin\alpha\cos\alpha,$$
$$\therefore\ 1:\cos 2\theta=a^2(b^2+c^2)^2\cos^2\alpha+b^2(a^2+c^2)^2\sin^2\alpha+c^2(a^2-b^2)^2\sin^2\alpha\cos^2\alpha$$
$$: a^2(b^2+c^2)^2\cos^2\alpha+b^2(a^2+c^2)^2\sin^2\alpha-c^2(a^2-b^2)^2\sin^2\alpha\cos^2\alpha.$$

(9) As in Art. 210, $l^2/a-m^2/b=0$ and $2lf/a-2mg/b=n$,

$$\text{whence } l : m : n=\tfrac{1}{2}\sqrt{a} : \pm\tfrac{1}{2}\sqrt{b} : f/\sqrt{a}\pm g/\sqrt{b},$$
$$\therefore\ \cos\theta\sqrt{[\{\tfrac{1}{4}(a+b)+f^2/a+g^2/b\}^2-4f^2g^2/ab]}=\tfrac{1}{4}(a-b)+h,$$

and $\{\tfrac{1}{4}(a-b)+h\}^2\tan^2\theta=\tfrac{1}{4}ab+\tfrac{1}{2}(a+b)(f^2/a+g^2/b)-\tfrac{1}{2}(a-b)h.$

(10) As in Art. 210,

$$mn+nl+lm=0 \text{ and } l(g+h)+m(h+f)+n(f+g)=0,$$
$$\therefore\ l_1l_2 : m_1m_2 : n_1n_2 : l_1m_2+l_2m_1 : l_1m_2-l_2m_1$$
$$=(h+f)(f+g):(f+g)(g+h):(g+h)(h+f):-2h(f+g):\sqrt{(8a^2)}(f+g),$$
$$\text{hence } \cos^2\theta : \sin^2\theta : 1=(r^2-6a^2)^2 : 8a^2(2r^2-4a^2) : (r^2+2a^2)^2.$$

(11) Shew as above that $\cos\theta:\sin\theta=a+b+c-r^2:2\sqrt{(-abc/p^2)}$

$$\text{and } \lambda^2+(a+b+c-r^2)\lambda+abc/p^2=0,$$
$$\text{hence } \cos^2\theta : \sin^2\theta : 1=(\lambda_1+\lambda_2)^2 : -4\lambda_1\lambda_2 : (\lambda_1-\lambda_2)^2.$$

(12) Equations of the shortest distance of generators corresponding to eccentric angles α and $\alpha+d\alpha$ being $(x-f)/l=(y-g)/m=z/n$,

$$l\sin\alpha-m\cos\alpha+n=0, \text{ and } l\cos\alpha+m\sin\alpha=0,$$
$$\text{so that } l : m : n=\sin\alpha : -\cos\alpha : -1,$$

also, where the lines meet the generator (α), $f-z\sin\alpha=a\cos\alpha+z\sin\alpha$ and $g+z\cos\alpha=a\sin\alpha-z\cos\alpha$, $\therefore\ f\cos\alpha+g\sin\alpha=a$, and, by the generator $(\alpha+d\alpha)$, $f\sin\alpha-g\cos\alpha=0$, hence for the shortest distance $(x/a-\cos\alpha)/\sin\alpha=(y/a-\sin\alpha)/-\cos\alpha=-z/a$.

(13) If $(x-\xi)/l=(y-\eta)/m=(z-\zeta)/n=r$ be the equations of a generator through (ξ, η, ζ),

$$a\xi^2+b\eta^2+c\zeta^2=1 \text{ and } a\xi l+b\eta m+c\zeta n=0,$$

hence, for any point in either generator, $a\xi x+b\eta y+c\zeta z=1$, a plane containing both, which is perpendicular to a generator in direction (λ, μ, ν), so that $a\lambda^2+b\mu^2+c\nu^2=0$ becomes $a^3\xi^2+b^3\eta^2+c^3\zeta^2=0$ (1).

Also, if the generators be at right angles, Art. 212 Cor.

$$\xi^2+\eta^2+\zeta^2=(a^{-1}+b^{-1}+c^{-1})(a\xi^2+b\eta^2+c\zeta^2),$$

$\therefore$ by (1), $a(b^{-1}+c^{-1})/a^3=\ldots$, or $(b+c)/a=(c+a)/b=(a+b)/c$,

$\therefore a+b+c=0$ unless $a=b=c$.

(14) From the symmetry the line of shortest distance passes through and is perpendicular to the axis of z; take as its equations $x/l=y/m=r$, $z=h$ (1), and for those of the generators

$$(x/a-\cos\alpha)/\sin\alpha=(y/b-\sin\alpha)/-\cos\alpha=z/c;$$

$$\therefore\ la\sin\alpha-mb\cos\alpha=0 \quad (2),$$

$$\text{also } lr/a=\cos\alpha+\sin\alpha\,h/c,\quad mr/b=\sin\alpha-\cos\alpha\,h/c,$$

$$\therefore\ ma(\cos\alpha+\sin\alpha\,h/c)-lb(\sin\alpha-\cos\alpha\,h/c)=0,$$

$$\text{and by (2) } ma(la+mb\,h/c)-lb(mb-la\,h/c)=0,$$

$$\text{by (1) } xy(a^2-b^2)+(x^2+y^2)zab/c=0$$

for one system of generators.

(15) A plane through the eye and any generating line must intersect the hyperboloid in another straight line, since the section is of the second degree.

Let (f, g, h) be the position of the eye E on the hyperboloid, (ξ, η, ζ) a point P at which the generating lines appear to be perpendicular, so that the planes containing them and the eye are at right angles, but $a\xi x+b\eta y+c\zeta z=1$ is a plane containing the generators through P, as in prob. 13;

$\therefore ax^2+by^2+cz^2-1+\rho(afx+bgy+chz-1)(a\xi x+b\eta y+c\zeta z-1)=0$ is the equation of a conicoid containing the four generators through E and P, which, since $af^2+\ldots=1$ and $a\xi^2+\ldots=1$, may be put in the form

$$a(x-f)^2+2af(x-f)+\ldots+\rho\{af(x-f)+\ldots\}\{a\xi(x-f)+\ldots+\sigma)$$

where $\sigma=a\xi f+b\eta g+c\zeta h-1$. If this coincoid be two perpendicular planes through E, $2+\rho\sigma=0$, and the sum of the coefficients of $(x-f)^2$, $(y-g)^2$ and $(z-h)^2$ must vanish.

(16) Take the axes and generators as in Art. 209; the angular points not on the hyperboloid are (a, b, c) and $(-a, -b, -c)$, $ayz+bzx+cxy+abc$ has values for either of these points, and for the centre, in the ratio 4 : 1, and this ratio is the same when the axes are transformed so that the equation becomes

$$x^2/a^2+y^2/b^2-z^2/c^2=1,\ \therefore\ 1-x^2/a^2-y^2/b^2+z^2/c^2 : 1 :: 4 : 1.$$

XIX.

(1) Two circular sections pass through the fixed point, and any sphere which passes through *either* of these sections intersects the conicoid in another plane section.

(2) The circular sections of $ax^2+by^2+cz^2=1$ which pass through (f, g, h) have the equation $(a-b)(x-f)^2=(b-c)(z-h)^2$, and the sphere containing both has equation

$$b(x^2+y^2+z^2)+2(a-b)fx-2(b-c)hz+\ldots=0,$$

the centre is $\{-(a-b)f/b,\ 0,\ (b-c)h/c\}$, and when the radii of the circular sections are equal, the centre of the sphere is equidistant from the two planes of section, $\therefore f=0$ or $h=0$.

Geometrically, the diameter of any circular section is a chord of the elliptic section in the plane of zx, and two equal chords must intersect in one of the axes.

(3) Let (l, m, n) be the direction of a normal to the plane, $\therefore l^2(a^2-d^2)+m^2(b^2-d^2)+n^2(c^2-d^2)=0$, $\therefore l^2a^2+m^2b^2+n^2c^2=d^2$, or the area is constant, Art. 237.

(4) Let (ξ, η, ζ) be the centre of a plane section through (f, g, h) whose equation is $l(x-f)+m(y-g)+n(z-h)=0$, $\therefore l(\xi-f)+\ldots=0$, and, by Art. 234, $a\xi/l=b\eta/m=c\zeta/n$; $\therefore a\xi(\xi-f)+b\eta(\eta-g)+c\zeta(\zeta-h)=0$ gives the locus.

(5) In the change the circles move in their own planes, the centre of the circle, whose distance from xy is z and radius $=(t^2z^2+a^2)^{\frac{1}{2}}$, moves to a point $(lz/n,\ mz/n,\ z)$.

(6) Take $y^2/b+z^2/c=2x$ for the paraboloid, $b>c$, and let C be the constant product of the radii R, R' of the cyclic sections through the point $(\xi, 0, \zeta)$, whose diameters are chords of the parabola $z^2=2cx$; for these chords $x=\pm mz+\alpha$, where $cm^2=b-c$; and if z_1, z_2 be the roots of $z^2=2c(mz+\alpha)$, $(z_2-z_1)^2=4m^2c^2+4.2c\alpha$, and $4R^2=(1+m^2)(z_2-z_1)^2$, hence $R^2=c(b-c+2\xi-2m\zeta)$, and $R'^2=c(b-c+2\xi+2m\zeta)$; $\therefore$ $C^2/c^2=(b-c+2\xi)^2-4m^2\zeta^2$, whose asymptotes are parallel to $\xi=\pm m\zeta$.

(7) Let $S=0$ be the equation of the sphere, $S-kL^2=0$ that of the paraboloid (or hyperboloid) of revolution, touched by the sphere along the plane $L=0$; P is a point in the section by the tangent plane to the sphere at H, PN, PM perpendiculars to the plane $L=0$ and its intersection MD with the tangent plane. PH is a tangent to the sphere and $PH^2 \propto S \propto L^2 \propto PN^2 \propto PM^2$; $\therefore H$ is the focus and MD the directrix.

Geometrically, for a hyperboloid of one sheet including the paraboloid. Let P be a point in the section, S, H the points of contact with the two spheres, QPR a generating line through P meeting the two circles of contact with the hyperboloid in Q and R; QPR is constant for all positions of P, but by equality of tangents from a point to a sphere, $SP=PQ$, $HP=PR$, $SP+PH=QR$. Given by Dallas, King's College.

(8) Let α, $\pi-\alpha$ be the inclinations to the plane of xy of the cyclic planes $x'Oy$, $z'Oy$ of the ellipsoid $ax^2+by^2+cz^2=1$ (1); for transforming we have $x=(x'-z')\cos\alpha$, $z=(x'+z')\sin\alpha$,

$$\therefore\ (a\cos^2\alpha+c\sin^2\alpha)(x'^2+z'^2)+by^2+2x'z'(c\sin^2\alpha-a\cos^2\alpha)=1,$$

$$\text{but } \sin^2\alpha:\cos^2\alpha:1=b-a:c-b:c-a,$$

$$\therefore\ (c-a)b(x'^2+y^2+z'^2)+\{b(c+a)-2ac\}\,2x'z'=c-a.$$

If $\alpha=\frac{1}{4}\pi$, $2b=a+c$; write $\sqrt{\frac{1}{2}}(x'-z')$ and $\sqrt{\frac{1}{2}}(x'+z')$ for x and z in (1).

(9) The projections on the plane xy will be parabolic.

For the first surface, $nxy-(x+y)(lx+my-p)=na^2$;

$$\therefore\ 4lm=(l+m-n)^2,\quad l+m-n=\pm2\sqrt{lm}.$$

For the second, $(lx+my)^2+2n(x+y)(lx+my)+n^2(x-y)^2=0$ has equal roots; $\therefore (l+n)^2(m+n)^2-\{lm+(l+m)n-n^2\}^2=0$.

XX.

(1) Eliminating y, we have $\sqrt{(a^2-b^2)}x/a\pm\sqrt{(b^2-c^2)}z/c=\sqrt{(a^2-c^2)}$, these are the equations of two planes which meet the ellipsoid where $y^2/b^2+\{\sqrt{(b^2-c^2)}\,x/a\mp\sqrt{(a^2-b^2)}\,z/c\}^2=0$, that is, in two indefinitely small circles.

(2) For a cyclic section, $\sqrt{(a^2-c^2)}\,y-\sqrt{(a^2+c^2)}\,z=\alpha$, and if (ξ,η,ζ) be the centre, (λ,μ,ν) the direction of any radius r, as in Art. 234, $\mu\sqrt{(a^2-c^2)}-\nu\sqrt{(a^2+c^2)}=0$, $\mu\eta-\nu\zeta=0$, $\xi=0$, and $\eta\sqrt{(a^2-c^2)}-\zeta\sqrt{(a^2+c^2)}=\alpha$, $\therefore \eta/\sqrt{(a^2-c^2)}=\zeta/\sqrt{(a^2+c^2)}=\alpha/-2c^2$, also $\{\lambda^2/a^2+(\mu^2-\nu^2)/c^2\}\,r^2=1-(\eta^2-\zeta^2)/c^2$, whence $r^2=a^2+\eta^2+\zeta^2$, the equation of the corresponding sphere is $x^2+y^2+z^2-2\eta y-2\zeta z=a^2$, and the radical plane of the spheres is $y\sqrt{(a^2-c^2)}+z\sqrt{(a^2+c^2)}=0$.

(3) Let the equation of two cyclic planes be

$$\{\sqrt{(a^2-b^2)}x/a-\sqrt{(b^2-c^2)}z/c+\alpha\}\{\sqrt{(a^2-b^2)}x/a+\sqrt{(b^2-c^2)}z/c+\alpha'\}=0;$$

that of the sphere containing the two circular sections is

$$b^2(x^2/a^2+y^2/b^2+z^2/c^2-1)+\{\sqrt{(a^2-b^2)}x/a-\ldots\}\{\sqrt{(a^2-b^2)}x/a+\ldots)=0,$$

or $x^2+y^2+z^2+(\alpha+\alpha')\sqrt{(a^2-b^2)}x/a+(\alpha-\alpha')\sqrt{(b^2-c^2)}z/c-b^2+\alpha\alpha'=0$, coordinates of the centre are

$$\xi=-\tfrac{1}{2}(\alpha+\alpha')\sqrt{(a^2-b^2)}/a,\quad \zeta=-(\alpha-\alpha')\sqrt{(b^2-c^2)}/c,$$

$$\text{and } m^2b^2=\xi^2+\zeta^2+b^2-\{\xi^2a^2/(a^2-b^2)-\zeta^2c^2/(b^2-c^2)\}.$$

(4) Using the equation of Art. 250, the coordinates of the centre are $-\frac{1}{2}(k+k')\sqrt{(a-b)/b}$, and $-\frac{1}{2}(k-k')\sqrt{(b-c)/b}$; and, if the centre is on the plane

$$\sqrt{(a-b)}\,x-\sqrt{(b-c)}\,z-k=0,\quad (a+c)k+(a-c)k'=0,$$

the line of intersection of the cyclic planes is

$$a\sqrt{(a-b)}\,x-c\sqrt{(b-c)}\,z=0.$$

If the centre of the sphere lie on the second plane the line of intersection will be $a\sqrt{(a-b)}\,x+c\sqrt{(b-c)}\,z=0$.

(5) By Art. 237, the square of the difference is

$$\{l^2(b+c)+m^2(c+a)+n^2(a+b)\}^2-4(l^2bc+m^2ca+n^2ab)^2,$$

which, by eliminating m^2, can be reduced to

$$\{c-a-(b-a)l^2-(c-b)n^2\}^2-4(b-a)(c-b)l^2n^2,$$

and if θ be the inclination to a cyclic plane,

$$(c-a)\sin^2\theta=c-a-\{l\sqrt{(b-a)}\pm n\sqrt{(c-b)}\}^2.$$

(6) Let $z\sqrt{(c-b)}+x\sqrt{(b-a)}=0$ be the cyclic plane common to the conicoid and the paraboloids; $z\sqrt{(c-b)}-x\sqrt{(b-a)}=\alpha$ the other cyclic plane common to the paraboloid whose vertex is $(\xi, 0, \zeta)$ and the conicoid. Since there is no term in x^2 in this paraboloid, its equation is

$$(b-a)(ax^2+by^2+cz^2-1)$$
$$+a\{z\sqrt{(c-b)}+x\sqrt{(b-a)}\}\{z\sqrt{(c-b)}-x\sqrt{(b-a)}-\alpha\}=0,$$

or $b(b-a)y^2+b(c-a)z^2-a\alpha\{z\sqrt{(c-b)}+x\sqrt{(b-a)}\}-(b-a)=0,$ comparing this with the other form of the equation

$$b(b-a)y^2+b(c-a)(z-\zeta)^2-lb(b-a)(x-\xi)=0,$$

$2b(c-a)\zeta=a\alpha\sqrt{(c-b)},\ lb\sqrt{(b-a)}=a\alpha,\ b(c-a)\zeta^2+lb(b-a)\xi=-(b-a),$ whence both results may be derived.

(7) Since $c-b=b-a$, the circular sections are $x\pm z=0$, and, by Art. 59, the corresponding cylinders are

$$b^{-1}=x^2+y^2+z^2-\tfrac{1}{2}(x\pm z)^2, \text{ or } 4=(a+c)\{(x\mp z)^2+2y^2\},$$

and for the plane sections

$$4(ax^2+cz^2)=(a+c)(x\mp z)^2 \text{ or } (x\pm z)\{(3a-c)x\pm(3c-a)z\}=0.$$

The area of the second section is $\pi(l^2bc+n^2ab)^{-\frac{1}{2}}$, Art. 237, where $l^2:n^2:1:l^2bc+n^2ab=(3a-c)^2:(3c-a)^2:8\{2(a^2+c^2)-3b^2\}:b(a+c)^3.$

(8) Take (λ, μ, ν) for the direction of any radius vector r of the section, so that $a\mu\nu+b\nu\lambda+c\lambda\mu+abc/r^2=0$, and $l\lambda+m\mu+n\nu=0$; eliminate ν, and, for the reason given in Art. 237, make the roots of the quadratic in $\lambda:\mu$ equal.

(9) If (λ, μ, ν) be the direction and r the length of a semi-axis of the section by a plane $lx+my+nz=0$, by (4) of Art. 237,

$$l/\lambda : m/\mu : n/\nu = ar^2-1 : br^2-1 : cr^2-1;$$
$$\therefore\ (b-c)\,l/\lambda+(c-a)\,m/\mu+(a-b)\,n/\nu=0;$$
$$\text{also, } l\alpha+m\beta+n\gamma=0, \text{ and } l\lambda+m\mu+n\nu=0,$$
$$\therefore\ l:m:n=\beta\nu-\gamma\mu:\gamma\lambda-\alpha\nu:\alpha\mu-\beta\lambda;$$

and, for any point in the cone, $x/\lambda=y/\mu=z/\nu$.

(10) Let $x/\lambda=y/\mu=z/\nu=r$ be the equation of a line in the plane, where it meets the surface,

$$(a\lambda^2+\ldots+2a'\mu\nu+\ldots)r^2+Ar+B=0;$$

and when r is infinite the directions are given by the equations

$$a\lambda^2+\ldots+2a'\mu\nu+\ldots=0, \text{ and } l\lambda+m\mu+n\nu=0; \quad (1)$$

as in Art. 26, eliminate ν, and the resulting quadratic gives

$$\lambda_1\lambda_2 : \mu_1\mu_2 = cm^2-2a'mn+bn^2 : an^2-2b'nl+cl^2,$$

whence the condition for the rectangular hyperbola.

For the parabola the directions are coincident and the roots of the quadratic are equal.

With the given relations the equation of the surface becomes

$$a'b'c'(x/a'+y/b'+z/c')^2+2a''x+\ldots+d=0,$$

and the directions are coincident for all values of l, m, and n, the surface being a parabolic cylinder.

(11) The area of the section by a plane $lx+my+nz=0$ is $\pi abc/\varpi$, where $\varpi^2=l^2a^2+m^2b^2+n^2c^2$, also $lx'+my'+nz'=0$, and ϖ_1, ϖ_2 the maximum and minimum values of ϖ are the roots of

$$x'^2/(\varpi^2-a^2)+y'^2/(\varpi^2-b^2)+z'^2/(\varpi^2-c^2)=0,$$

which, by Art. 237, (3) proves the first part.

Also, $\varpi_1{}^2\varpi_2{}^2=(b^2c^2x'^2+\ldots)/(x'^2+y'^2+z'^2)$, if (x', y', z') lie on the ellipsoid and the sphere $x^2+y^2+z^2=d^2$, $d\varpi_1\varpi_2=abc$.

(12) For the central sections

$$x^2+y^2+z^2-r^2-r^2(ayz+bzx+cxy-1)\equiv(lx+my+nz)(xl^{-1}+ym^{-1}+zn^{-1});$$

$$\therefore\ mn^{-1}+nm^{-1}=-r^2a, \text{ \&c.} \quad (1),$$

$$-r^6abc=(mn^{-1}+nm^{-1})(nl^{-1}+ln^{-1})(lm^{-1}+ml^{-1})$$

$$=m^2n^{-2}+n^2m^{-2}+n^2l^{-2}+l^2n^{-2}+l^2m^{-2}+m^2l^{-2}+2=r^4(a^2+b^2+c^2)-6+2.$$

Also, by (1), $(m^2+n^2)\,l/a=-lmnr^2=\ldots$.

(13) Let (ξ, η, ζ) be the focus, $x=\xi$, $(y-\eta)/\mu=(z-\zeta)/\nu=r$ equations of the latus rectum of the parabolic section; at the ends of the latus rectum $b(\eta+\mu r)^2+c(\zeta+\nu r)^2=2\xi$, and the coefficient of r vanishes, $\therefore b\eta\mu+c\zeta\nu=0$; if (ξ', η, ζ) be the vertex $b\eta^2+c\zeta^2=2\xi'$, and the semi-latus rectum $=r=2(\xi-\xi')=(b\mu^2+c\nu^2)r^2$;

$$\therefore\ (b\mu^2+c\nu^2)(2\xi-b\eta^2-c\zeta^2)=\mu^2+\nu^2.$$

XXI.

(1) Since they are tangent planes to the enveloping cone, their intersections with a plane are tangents to the conic section.

(2) The tangent plane at (ξ, η, ζ) is $x\xi/a^2+y\eta/b^2+z\zeta/c^2=1$, and when $y=0$, $\xi/a^2 : \zeta/c^2 = c\sqrt{(b^2-c^2)} : \pm a\sqrt{(a^2-b^2)}$;

$$\therefore\ c\sqrt{(a^2-b^2)}\,\xi\mp a\sqrt{(b^2-c^2)}\,\zeta=0.$$

(3) A tangent plane to $\alpha x^2+\ldots=1$ is $lx+my+nz=\sqrt{(l^2/\alpha+\ldots)}$, and if (ξ, η, ζ) be the centre of the section made by this plane, $(l\xi+m\eta+n\zeta)^2=l^2/\alpha+m^2/\beta+n^2/\gamma$, and $l:m:n=a\xi:b\eta:c\zeta$, $\therefore$ &c.

(4) $l(x-f)+m(y-g)+n(z-h)=0$ is the equation of any plane passing through a point $P(f, g, h)$ on the conicoid $ax^2+by^2+cz^2=1$; take a point $Q(f+f', g+g', h+h')$ in the conicoid near to P, QM perpendicular to the plane $=lf'+mg'+nh'$, and $aff'+bgg'+chh'=-\frac{1}{2}(af'^2+bg'^2+ch'^2)$; $\therefore$ if $l:m:n=af:bg:ch$, QM will be of the second degree in the small quantities f', g', h', and is less than for any other plane through P.

(5) If $lx+my+nz=0$ be a tangent plane to the cone,

$$\therefore\ l^2(b+c)+m^2(c+a)+n^2(a+b)=0, \text{ Art. } 257;$$

and if (λ, μ, ν) be the direction of an asymptote of the section of the surface, $a\lambda^2+b\mu^2+c\nu^2=0$ and $l\lambda+m\mu+n\nu=0$; shew that $\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2=0$.

(6) Let (λ, μ, ν) be the direction of the axis of the cylinder, the equation of the plane of the curve of contact is $a\lambda x+b\mu y+c\nu z=0$, see Arts. 262 and 267; shew by Art. 237 that

$$a^2\lambda^2bc+b^2\mu^2ca+c^2\nu^2ab=ac(a^2\lambda^2+b^2\mu^2+c^2\nu^2),$$

and thence that the two planes are $a(b-a)x^2=c(c-b)z^2$.

(7) The line $y=\beta(x-a)$, $z=\gamma(x+a)$ touches the sphere $x^2+y^2+z^2=c^2$, eliminate y and z and make the roots of the quadratic in x equal, $(1+\beta^2+\gamma^2)\{(\beta^2+\gamma^2)a^2-c^2\}=a^2(\beta^2-\gamma^2)^2$, the equation of the locus is

$$\{y^2(x+a)^2+z^2(x-a)^2\}(a^2-c^2)+4a^2y^2z^2=c^2(x^2-a^2)^2,$$

when $c=a$, $x^2\pm 2yz=a^2$, or transformed $x^2\pm y^2\mp z^2=a^2$.

(8) Let (f, g, h) be the point P, p the perpendicular from the centre on the tangent, $x^2/a^2+\ldots=1$ the ellipsoid,

$$a^2(x-f)/f=b^2(y-g)/g=c^2(z-h)/h=pr,$$

and if $x=0$, $r=-PG_1$, $\therefore\ p.PG_1=a^2$, and PG_1 varies as the area given, which is $\pi abc/p$.

(9) The shadow is the section by the plane $z=-c$ of the cylindrical envelope

$$(\lambda^2/a^2+\mu^2/b^2+\nu^2/c^2)(x^2/a^2+y^2/b^2+z^2/c^2-1)=(\lambda x/a^2+\mu y/b^2+\nu z/c^2)^2,$$

the direction of whose axis is (λ, μ, ν);

$$\therefore\ (\lambda^2/a^2+\ldots)(x^2/a^2+y^2/b^2)=(\lambda x/a^2+\mu y/b^2-\nu/c)^2$$

is the equation of a circle, $\therefore\ \lambda\mu=0$, let $\lambda=0$, and equate the coefficients of x^2 and y^2, shew that $\nu^2/\mu^2=c^2/(a^2-b^2)$.

(10) For the point whose locus is required

$$a^2(x-f)/f=b^2(y-g)/g=c^2(z-h)/h=-pr=-m^2;$$

$$\therefore\ (a^2-m^2)f=a^2x \text{ and } f^2/a^2+\ldots=1.$$

If $m=b$, $y=0$, and the locus is the limit of a very flat ellipsoid bounded by the ellipse, $a^2x^2/(a^2-b^2)^2+c^2z^2/(b^2-c^2)^2=1$.

(11) Let $lx+my+nz=p$ be the equation of the plane,
$$\therefore\ lx_1+my_1+nz_1=p,\ lx_2+\ldots=p \text{ and } lx_3+\ldots=p;$$
$\therefore$, by (3), Art. 276, $l=ap(x_1+x_2+x_3)$, &c. Also, by (2), Art. 276, $lx_1+my_1+nz_1=p$, and $l^2/a+m^2/b+n^2/c=3p^2$.

(12) The centre of gravity of the triangle is that of three equal masses placed at the extremities of the diameters, and
$$a\{\tfrac{1}{3}(x_1+x_2+x_3)\}^2+b\{\tfrac{1}{3}(y_1+y_2+y_3)\}^2+c\{\tfrac{1}{3}(z_1+z_2+z_3)\}^2=\tfrac{1}{3}.$$
Also, for the second locus, $ax_1x+by_1y+cz_1z=1$, $ax_2x+\ldots=1$, $ax_3x+\ldots=1$, square and add and use (3), Art. 276.

(13) The product of the perpendiculars from (x_1, y_1, z_1), and $(-x_1, -y_1, -z_1)$ upon the tangent plane at (f, g, h)
$$=p^2\{1-(afx_1+bgy_1+chz_1)^2\},$$
and by (3), Art. 276, the sum required $=p^2(3-af^2-bg^2-ch^2)=2p^2$.

(14) By Art. 258, for the two asymptotes through (ξ, η, ζ), $a\lambda^2+b\mu^2+c\nu^2=0$ and $a\xi\lambda+b\eta\mu+c\zeta\nu=0$, and the condition of perpendicularity gives the equation of the cone.

(15) For any point in the normal at (x', y', z'),
$a^2(x/x'-1)=b^2(y/y'-1)=c^2(z/z'-1)=\rho$, and $a^2(f/x'-1)=\ldots=\sigma$;
$\therefore\ x/x'=\rho/a^2+1$ and $f/x'=\sigma/a^2+1$, and $f/(x-f)=(\sigma-a^2)/(\rho-\sigma)$.

(16) As in Art. 272, if (ξ, η, ζ) be the middle point of a chord in direction (λ, μ, ν) of the conicoid $ax^2+by^2+cz^2=1$,
$$a\lambda\xi+b\mu\eta+c\nu\zeta=0, \quad (1)$$
and at its ends $x=\xi\pm\lambda r$, &c.; at the intersection of the normals $(a\rho+1)(\xi+\lambda r)=(a\rho'+1)(\xi-\lambda r)$, Art. 270;
$$\therefore\ a\xi(\rho-\rho')/\lambda+r\{a(\rho+\rho')+2\}=0, \text{ \&c.,}$$
and, multiplying the three equations by $b-c$, $c-a$, $a-b$, and adding, $(b-c)a\xi/\lambda+(c-a)b\eta/\mu+(a-b)c\zeta/\nu=0$, this and (1) are the equations of the locus.

XXII.

(1) Let $lx+my+nz=0$ be a tangent plane to the cone, where $l^2/a+m^2/b+n^2/c=0$ (1); then at the foot of the perpendicular,
$$(x-\alpha)/l=\ldots=-l\alpha-m\beta-n\gamma=\{x(x-\alpha)+\ldots\}/0,$$
$$\therefore\ (x-\alpha)^2+\alpha(x-\alpha)+\ldots=0 \text{ and, by (1), } (x-\alpha)^2/a+\ldots=0.$$
Let this curve be plane, and let $(y-\beta)^2$ be eliminated,
$(x-\alpha)^2(b/a-1)+(\zeta-\gamma)^2(b/c-1)=\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)$ will give two planes, when $\beta=0$, $\alpha^2/(b/a-1)=\gamma^2(1-b/c)$; if $(x-\alpha)^2$ or $(z-\gamma)^2$ were eliminated the planes would be imaginary.

(2) Let $ax^2+\ldots=1$ and $=n^2$ be the two ellipsoids; where the enveloping cone, vertex (f, g, h) intersects the exterior

$$(n^2-1)^2=(afx+bgy+chz-1)^2, \text{ Art. } 265,$$

$\therefore$ $afx+bgy+chz=n^2$ or $-(n^2-2)$, the last will be a tangent plane to an ellipsoid $a'x^2+\ldots=1$ at a point (f', g', h'), if $af=(2-n^2)a'f'$, &c.,

$$\text{and } n^2=(2-n^2)^2(a'^2f'^2/a+\ldots)=n^2(a'f'^2+\ldots),$$

$$\therefore a'/a=b'/b=c'/c=n^2/(2-n^2)^2.$$

(3) Let (l, m, n) be the direction of the chord joining points whose coordinates are f, g, h, and $f+lr$, $g+mr$, $h+nr$ respectively, and let ϕ, ϕ' be the angles made with the normals at its extremities, p, p' the perpendiculars on the tangent planes from the centre. By Art. 269, afp, bgp, chp are the direction-cosines of the normal at (f, g, h); $\therefore \cos\phi/p=laf+mbg+nch$, similarly

$$\cos\phi'/p'=la(f+lr)+mb(g+mr)+nc(h+nr),$$

$$\text{also } (l^2a+m^2b+n^2c)r+2(laf+mbg+nch)=0;$$

$$\therefore \cos\phi/p=-\cos\phi'/p'.$$

(4) By Art. 271, the feet of the six normals lie in the two planes $lx/a+\ldots=1$ and $x/al+\ldots=-1$, and $al=\alpha$, $-a/l=\alpha'$, &c.

(5) By Art. 270, if f, g, h be coordinates of Q, x_r, y_r, z_r those of P_r, and (λ, μ, ν) the direction of OQ,

$ON_r=\lambda x_r+\mu y_r+\nu z_r$, $OP_r^2-OQ.ON_r=x_r(x_r-f)+\ldots=\rho_r$, Art. 270, and by the sextic equation $\Sigma(\rho_r)=2(a^2+b^2+c^2)$.

(6) Let the chord be normal at (f, g, h), the other extremity being $(f+\lambda n, g+\mu n, h+\nu n)$, where $\lambda=-afp$, &c., and this is on the surface, $\therefore n(a\lambda^2+b\mu^2+c\nu^2)+2(a\lambda f+b\mu g+c\nu h)=0$;

$\therefore a^3f^2+\ldots=2n^{-1}p^{-3}$, $a^2f^2+\ldots=p^{-2}$, $af^2+\ldots=1$, $f^2+\ldots=r^2$, multiply the last three by $-(a+b+c)$, $bc+ca+ab$, and $-abc$, respectively, and add.

(7) As in Art. 276, $x_1^2+x_2^2+x_3^2=a^2$, $y_1^2+\ldots=b^2$, $z_1^2+\ldots=c^2$.

$$\therefore 3(x_1^2+y_1^2+z_1^2)=(a^2+b^2+c^2)(x_1^2/a^2+y_1^2/b^2+z_1^2/c^2),$$

and, for any point in the corresponding conjugate diameter,

$$x/x_1=y/y_1=z/z_1; \quad \therefore (2a^2-b^2-c^2)x^2/a^2+\ldots=0. \quad (1)$$

By (2), Art. 276, $xx_3/a^2+yy_3/b^2+zz_3/c^2=0$ is a plane through (x_1, y_1, z_1) and (x_2, y_2, z_2), which touches $x^2/\alpha+\ldots=0$, if $\alpha x_3^2/a^4+\beta y_3^2/b^4+\gamma z_3^2/c^4=0$, $\therefore$, by (1), $\alpha=a^2(2a^2-b^2-c^2)$, &c.

(8) Three perpendicular tangent planes to the two conicoids are

$$l_1x+m_1y+n_1z=\sqrt{(l_1^2a+m_1^2b+n_1^2c)} \text{ or } -(m_1^2b+n_1^2c)/2l_1,$$

$$l_2x+\ldots=\ldots, \text{ and } l_3x+\ldots=\ldots.$$

Square and add for the first locus.

Multiply by $2l_1$, $2l_2$, $2l_3$ and add for the second.

(9) Transforming to the three tangents through (ξ, η, ζ) as axes, $x = \xi + lx' + l'y' + l''z'$, $y = \eta + \&c.$, in the transformed equation let $y' = 0$ and $z' = 0$, $(\xi + lx')^2/a + (\eta + mx')^2/b + (\zeta + nx')^2/c = 1$, has equal roots,

$\therefore (\xi^2/a + \eta^2/b + \zeta^2/c - 1)(l^2/a + m^2/b + n^2/c) = (l\xi/a + m\eta/b + n\zeta/c)^2$, adding to the corresponding equations

$$(\xi^2/a + \eta^2/b + \zeta^2/c - 1)(a^{-1} + b^{-1} + c^{-1}) = \xi^2/a^2 + \eta^2/b^2 + \zeta^2/c^2.$$

(10) By (1) and (2), Art. 210, the condition of perpendicularity of the generators through (f, g, h) is $(b + c)\,a^2f^2 + \ldots = 0$, (1)

$(f' - f)/af = (g' - g)/bg = (h' - h)/ch$, and $a(f'^2 - f^2) + \ldots = 0$;

$$\therefore a^2f(f' + f) + b^2g(g' + g) + c^2h(h' + h) = 0,$$

and $(f' - f)/af = -2(a^2f^2 + \ldots)/(a^3f^2 + \ldots) = -2/(a + b + c)$ by (1).

(11) By iv., Art. 268, if (l, m, n) be the direction of a generating line $(bg^2 + ch^2 - 2f)(bm^2 + cn^2) = (bgm + chn - l)^2$, and writing for l, m, n their values for three perpendicular generators, and adding $(bg^2 + ch^2 - 2f)(b + c) = b^2g^2 + c^2h^2 + 1$ or $bcg^2 + bch^2 = 2f(b + c) + 1$, (1) which gives the locus of the vertex (f, g, h). The equation of the polar plane of the vertex is $bgy + chz = x + f$, (2). The tangent plane at (f', g', h') to $b'y^2 + c'z^2 = 2x + \alpha$ is $b'g'y + c'h'z = x + f' + \alpha$, which coincides with (2) if $b'g' = bg$, $c'h' = ch$, and $f' + \alpha = f$; also $b'g'^2 + c'h'^2 = 2f' + \alpha$, or $b^2g^2/b' + c^2h^2/c' = 2f - \alpha$, which is the same as (1) if $cb'/b = bc'/c = b + c = -\alpha^{-1}$, $\therefore$ the polar plane touches the paraboloid $(b + c)(by^2/c + cz^2/b) = 2x - (b + c)^{-1}$.

(12) For the normal at (f, g, h), $x = f + af\sigma$, for that at (ξ, η, ζ), $x = \xi + a\xi\rho$, if these intersect, $\xi - f = af\sigma - a\xi\rho$, &c.;

$$\therefore bc\{g(\zeta - h) - h(\eta - g)\}(\xi - f) + \ldots = 0.$$

(13) For the two tangent planes,

$$lx + my + nz = 0, \text{ and } l^2a + m^2b + n^2c = 0, \text{ Art. 257},$$

the equation of the cone is the condition of perpendicularity.

(14) Let R be the semi-diameter in direction (λ, μ, ν), so that $a\lambda^2 + b\mu^2 + c\nu^2 = R^{-2}$, $(\lambda r, \mu r, \nu r)$ and $(\lambda r', \mu r', \nu r')$ the vertices of the cone, $\therefore$ writing these for (f, g, h) in the envelope, Art. 265, the cones intersect in the parallel planes

$$\{(a\lambda x + \ldots)r - 1\}\sqrt{(r'^2 - R^2)} \pm \{(a\lambda x + \ldots)r' - 1\}\sqrt{(r^2 - R^2)} = 0,$$

if p_1, p_2 be the perpendiculars from the centre, shew that

$$p_1p_2 = (a^2\lambda^2 + b^2\mu^2 + c^2\nu^2)^{-2}R^{-2} = p^2.$$

XXIII.

(1) For the normal at (x, y, z) passing through (f, g, h), $(x - f)/ax = (y - g)/by = (z - h)/cz = \sigma^{-1}$ suppose, then substituting in $ax^2 + by^2 + cz^2 = 0$, since $x(\sigma - a) = \sigma f$, &c., (1)

$af^2(\sigma-b)^2(\sigma-c)^2+\ldots=(af^2+bg^2+ch^2)(\sigma-\sigma_1)(\sigma-\sigma_2)(\sigma-\sigma_3)(\sigma-\sigma_4)$, (2)
$\sigma_1, \sigma_2, \sigma_3, \sigma_4$ being the values of σ for A, B, C, D.

Let the equation of plane ABC be $\alpha x+\beta y+\gamma z=1$,
$\therefore af\sigma_1/(\sigma_1-a)+\ldots=1$, $af\sigma_2/(\sigma_2-a)+\ldots=1$, and $af\sigma_3/(\sigma_3-a)+\ldots=1$, subtracting,
$afa/(\sigma_1-a)(\sigma_2-a)+\ldots=0$, and $afa/(\sigma_1-a)(\sigma_3-a)+\ldots=0$, (3)
and, by (2), $(af^2+\ldots)(\sigma_1-a)(\sigma_2-a)(\sigma_3-a)(\sigma_4-a)=af^2(a-b)^2(a-c)^2$,
$\therefore afa/(\sigma_1-a)(\sigma_2-a)(\sigma_3-a) \propto a(b-c)^2(\sigma_4-a)/f \propto a(b-c)^2/x_4$, by (1),
let α', β', γ' be written for this and similar expressions;
$\therefore$, by (3), $\alpha'(\sigma_3-a)+\beta'(\sigma_3-b)+\gamma'(\sigma_3-c)=0$ and $\alpha'(\sigma_2-a)+\ldots=0$,
$\therefore \alpha'+\beta'+\gamma'=0$, $\alpha'a+\beta'b+\gamma'c=0$, and $\alpha'/(b-c)=\beta'/(c-a)=\gamma'/(a-b)$,

$$\text{hence } x_4/\alpha(b-c)=y_4/\beta(c-a)=z_4/\gamma(a-b);$$
$$\therefore a(b-c)^2\alpha^2+b(c-a)^2\beta^2+c(a-b)^2\gamma^2=0.$$

(2) If the generators through P be $v=0$, $w=0$, and $v'=0$, $w'=0$, the equations of B and A will be of the form $vv'+ww'=0$ and $vv'+ww'+u^2=0$; also, for the conic S, $v=0$ and $ww'+u^2=0$, hence the generators meet the conic each in two coincident points.

(3) By Art. 231, all parabolic sections are parallel to tangent planes of the conical asymptote, which shews that $l^2/a+\ldots=0$, Art. 257, and the lines of contact with the cone determine the two coincident directions in which the parabolas pass off to infinity; so that if (λ, μ, ν) be the direction of the axis $a\lambda/l=b\mu/m=c\nu/n$, (1). Let (ξ, η, ζ) be the vertex of the parabolic section by the given plane, (λ', μ', ν') the direction of the chords bisected at right angles by the axis; $\therefore a\xi\lambda'+b\eta\mu'+c\zeta\nu'=0$, and $\lambda\lambda'+\mu\mu'+\nu\nu'=0$, or, by (1), $l\lambda'/a+m\mu'/b+n\nu'/c=0$, also $l\lambda'+m\mu'+n\nu'=0$, whence, by eliminating $l\lambda'$, $m\mu'$, and $n\nu'$, $(b^{-1}-c^{-1})a\xi/l+\ldots=0$.

(4) Using the method of Art. 271, the six feet of the normals from (ξ, η, ζ) lie on two planes $x/a+y/b+z/c\pm1=0$, if their coordinates satisfy the equation

$$(x/a+y/b+z/c)^2-1-(x^2/a^2+y^2/b^2+z^2/c^2-1)=0,$$

that is, if ξ, η, ζ can be chosen so that $ayz+bzx+cxy\equiv 0$, but since $U=0$, $V=0$, and $W=0$, substituting for yz, zx, and xy, we obtain $\{c\zeta/(c^2-a^2)-b\eta/(a^2-b^2)\}x+\ldots\equiv 0$;

$$\therefore a\xi(b^2-c^2)=b\eta(c^2-a^2)=c\zeta(a^2-b^2).$$

(5) The feet of the normals through (ξ, η, ζ) lie in the three cylinders, $(b-c)yz=b\eta z-c\zeta y$, $zx=(\xi-c)z+c\zeta$, and $xy=(\xi-b)y+b\eta$, (1) they also lie in two planes, one of which is $px+qy+rz=1$; also
$(px+qy+rz-1)(y/bq+z/cr-2/p)-y^2/b-z^2/c+2x$
$\equiv(q/cr+r/bq)yz+pzx/cr+pxy/bq-2q/py-2r/pz+2/p=0$

is true for all the feet, if (ξ, η, ζ) be taken so as to satisfy identically the equation $By + Cz + D = 0$, obtained by substituting for yz, zx, xy their values from (1). $B = 0$, $C = 0$, $D = 0$ with the given relation are equivalent to two equations in ξ, η, ζ.

(6) The equation of the enveloping cone is

$$(f^2/a^2+g^2/b^2+h^2/c^2-1)(x^2/a^2+y^2/b^2+z^2/c^2-1)=(fx/a^2+gy/b^2+hz/c^2-1)^2,$$

if $z = 0$, $(f^2/a^2+\ldots-1)(x^2/a^2+y^2/b^2-1) = (fx/a^2+gy/b^2-1)^2$. (1)

The equation of lines parallel to the asymptotes is

$$(g^2/b^2 + h^2/c^2 - 1)\, x^2/a^2 - 2fgxy/a^2b^2 + (f^2/a^2 + h^2/c^2 - 1)\, y^2/b^2 = 0,$$

which are at right angles if $f^2 + g^2 + (a^2 + b^2)(h^2/c^2 - 1) = 0$. (2)

If (ξ, η) be the centre of the curve (1),

$$(f^2/a^2+\ldots-1)\,\xi = f\,(f\xi/a^2 + g\eta/b^2 - 1), \text{ \&c.},$$

$$\frac{\xi}{f} = \frac{\eta}{g} = \frac{f\xi/a^2 + g\eta/b^2 - 1}{f^2/a^2 + g^2/b^2 + h^2/c^2 - 1} = \frac{f\xi/a^2 + g\eta/b^2}{f^2/a^2 + g^2/b^2} = \frac{1}{h^2/c^2 - 1};$$

$$\therefore \text{ by (2)}, \quad \sqrt{(\xi^2 + \eta^2)}\,\sqrt{(f^2 + g^2)} = a^2 + b^2.$$

(7) Let (ξ, η, ζ) be a point in the line, and

$$\lambda x + \mu y + \nu z = \sqrt{(\lambda^2/a + \mu^2/b + \nu^2/c)}$$

one of the tangent planes; $\therefore$ $l\lambda + m\mu + n\nu = 0$, eliminate ν and shew that $\lambda_1\lambda_2 : \mu_1\mu_2 : \nu_1\nu_2 = (n\eta - m\zeta)^2 - n^2/b - m^2/c$: &c.

(8) For the tangent plane $\lambda x + \mu y + \nu z = -(b\mu^2 + c\nu^2)/2\lambda$, Art. 268; as in (7),

$$\lambda_1\lambda_2 : \mu_1\mu_2 : \nu_1\nu_2 = bn^2 + cm^2 : 2n\,(n\xi - l\zeta) + cl^2 : 2n\,(m\xi - l\eta) + bl^2.$$

(9) As in (5), $(lx+my+nz)(y/mb+z/nc-2/l)-y^2/b-z^2/c+2x=0$, also, by the property of the tangent plane to the cone,

$$\tfrac{1}{2}(b-c)\,l^2 + bm^2 - cn^2 = 0,$$

eliminate l, m, n from the equations corresponding to $B = 0$, $C = 0$, $D = 0$.

(10) Referring to the generators through O as axes of x and y, the equation is $xy + (ax + by + cz + d)\,z = 0$, and the tangent plane at P is $(y + az)\,\xi + (x + bz)\,\eta + (ax + by + cz + d)\,\zeta + dz = 0$. At D, if on Ox, $(y+az)\,\xi = -dz$; at E, $(x+bz)\,\eta + dz = 0$, $\xi\eta$ is constant, $\therefore$ $(y+az)(x+bz) \propto z^2 = pz^2$; if (x_0, y_0, z_0) be the centre of the hyperboloid, $y_0 + az_0 = 0$, $x_0 + bz_0 = 0$, and $ax_0 + by_0 + cz_0 + d = 0$, shew that the polar of this centre with respect to this cone is the plane of xy.

(11) Let $(x_0, 0, z_0)$ be the umbilic U, θ the inclination of the tangent plane to the plane xy, and let ξ, η be coordinates, in the tangent plane, of any point of the section referred to U as origin. Then $x = x_0 + \xi\cos\theta$, $z = z_0 - \xi\sin\theta$.

Since $ax_0^2 + cz_0^2 = 1$, and $\tan\theta = x_0\sqrt{a}/z_0\sqrt{c}$,

$$ax^2 + cz^2 = 1 + \xi^2\,(a\cos^2\theta + c\sin^2\theta) = 1 + b\xi^2;$$

therefore the equation of the section of the enveloping cone is

$$b(af^2+bg^2+ch^2-1)(\xi^2+\eta^2)=(A+B\xi+bg\eta)^2,$$

where $A+B\xi+bg\eta=0$ is the line in which the plane of contact is cut by the tangent plane, and is the directrix of the section.

(12) Let $(0, \beta, \gamma)$ be the vertex, $\beta y/b^2+\gamma z/c^2=1$ is the plane of contact, intersecting the ellipsoid in the plane of yz, where

$$(\beta^2/b^2+\gamma^2/c^2)\,y^2-2\beta y+b^2(1-\gamma^2/c^2)=0,$$

of which y_1, y_2 are roots; at the centre of the section

$$y_0=\tfrac{1}{2}(y_1+y_2)=\beta/(\beta^2/b^2+\gamma^2/c^2);$$

and if a', b' be the semiaxes of the section, shew that

$$b'^2=\tfrac{1}{4}(y_1-y_2)^2(\beta^2c^4/\gamma^2b^4+1)$$

$$=b^2c^2(\beta^2/b^2+\gamma^2/c^2-1)(\beta^2/b^4+\gamma^2/c^4)/(\beta^2/b^2+\gamma^2/c^2)^2,$$

$$\text{and } a'^2/a^2=1-y_0^2/b^2-z_0^2/c^2=1-(\beta^2/b^2+\gamma^2/c^2)^{-1};$$

$$\therefore a'^4/(a'^2-b'^2)=a^4(\beta^2/b^2+\gamma^2/c^2+h\beta^2-k\gamma^2)/\{(a^2-c^2)\beta^2/b^2+(a^2-b^2)\gamma^2/c^2\}$$

which is constant, since $(1+hb^2)/(a^2-c^2)=(1-kc^2)/(a^2-b^2)$, *i.e.* the directrices are at a constant distance from the plane zy.

XXIV.

(1) Take the three confocals $x^2/a+\ldots=1$, $x^2/(a+k)+\ldots=1$ and $x^2/(a+k')+\ldots=1$; use the form of the tangent plane,

$$lx+my+nz=\sqrt{(l^2a+m^2b+n^2c)},$$

and shew that $x^2+y^2+z^2=a+b+c+k+k'$.

(2) In XXI (15) the coefficients b^2-c^2 &c. are the same for all the confocals.

(3) Take (f, g, h) the given point, then a'^2, a''^2, a'''^2 are the three roots of $f^2/\rho^2+g^2/(\rho^2-\beta^2)+h^2/(\rho^2-\gamma^2)=1$, so that

$$a'^2+a''^2+a'''^2=f^2+g^2+h^2+\beta^2+\gamma^2.$$

Add the three equations similar to $(lf+mg+nh)^2=a^2-m^2\beta^2-n^2\gamma^2$.

(4) Let $my+nz=1$, $x=0$ be the fixed line in yz, and let $x^2/(a+k)+\ldots=1$ be one of the confocals; a tangent plane at (ξ, η, ζ) meets the plane of yz in the line $\eta y/(b+k)+\zeta z/(c+k)=1$; $\therefore$ $\eta=m(b+k)$ and $\zeta=n(c+k)$, and for all the confocals $\eta/m-\zeta/n=b-c$. If $lx+my+nz=0$ be the plane of the second part, the three plane loci intersect in the line

$$\xi/l-a=\eta/m-b=\zeta/n-c.$$

(5) The foot of the normal at (f, g, h) is $(a-c)f/a$, $(b-c)g/b$, 0, and the polar with respect to $x^2/(a-c)+y^2/(b-c)=1$ is $xf/a+yg/b=1$.

(6) Let P, P' be (f, g, h) and (f', g', h'), the direction of PQ being (l, m, n); $\therefore f/f' = \sqrt{\{a/(a+k)\}}$ &c.,

$$lf/a + \ldots = 0 \text{ and } l^2/a + \ldots = 0;$$

$\therefore lf'/\sqrt{\{a(a+k)\}} + \ldots = 0$, but if $l' = l\sqrt{\{(a+k)/a\}}$ &c., then

$$l'f'/(a+k) + \ldots = 0,\ l'^2/(a+k) + \ldots = l^2/a + \ldots = 1,$$

$\therefore (l', m', n')$ is the direction of the generator through P', similarly for Q'.

Let (f_1, g_1, h_1) and (f_1', g_1', h_1') be Q and Q'; $\therefore f/f' = f_1/f_1'$, &c.,

$$P'Q^2 - PQ'^2 = (f' - f_1)^2 - (f - f_1')^2 + \ldots = \frac{k}{a}f^2 - \frac{k}{a}f_1^2 + \ldots = 0.$$

(7) Let (f, g, h) be a point on the ellipsoid $x^2/a + \ldots = 1$, (x, y, z) the corresponding point on $x^2/(a+k) + \ldots = 1$;

$$\therefore x^2/(a+k) = f^2/a = (x^2 - f^2)/k, \text{ \&c.},$$

the locus is the intersection of

$$ax^2/f^2 - by^2/g^2 = a - b \text{ and } ax^2/f^2 - cz^2/h^2 = a - c.$$

(8) Let $x^2/a + \ldots = 1$ and $x^2/(a+k) + \ldots = 1$ be the ellipsoid and hyperboloid, (ξ, η, ζ) the point on the sphere corresponding to (x, y, z); then $x^2/a(a+k) + \ldots = 0$, $\therefore \xi^2/(a+k) + \ldots = 0$.

XXV.

(1) By Art. 286. $f^2/a^2a'^2a''^2a'''^2 = 1/a^2(a^2 - b^2)(a^2 - c^2)$ &c.

(2) Let (l, m, n) be the direction of the normal at (ξ, η, ζ) to the confocal $x^2/(a+k) + \ldots = 1$; $\therefore l = \rho\xi/(a+k)$ &c.;

$$\therefore l\xi + m\eta + n\zeta = \rho,\ a + k = \rho\xi/l,\ b + k = \rho\eta/m, \text{ \&c.},$$

hence the locus is the section of the hyperbolic cylinder

$$(\xi/l - \eta/m)(l\xi + m\eta + n\zeta) = a - b$$

by the plane $(\xi/l - \eta/m)/(a-b) = (\xi/l - \zeta/n)(a-c)$, one asymptote of the section is $\xi/l = \eta/m = \zeta/n$ and the other is in the plane $l\xi + m\eta + n\zeta = 0$.

(3) The cylinder enveloping $u \equiv x^2/a + \ldots = 1$ is $(\lambda^2/a + \mu^2/b + \nu^2/c)(x^2/a + y^2/b + z^2/c - 1) = (\lambda x/a + \mu y/b + \nu z/c)^2$, or $L(u-1) = v^2$; make $r^2 = x^2 + y^2 + z^2$ a maximum or minimum, subject to the condition $\lambda x + \mu y + \nu z = 0$, and shew that

$$Lx(1 - a/r^2) = \lambda v, \text{ \&c.}; \therefore \lambda^2/(r^2 - a) + \ldots = 0,$$

whence $r^2 = \frac{1}{2}(b+c)\lambda^2 + \ldots \pm M$, where M depends only on the differences of a, b and c, and if α^2, β^2 be the two values of r^2, $\alpha^2 - \beta^2$ will be the same for any confocal. Also

$$r^2 - a = \tfrac{1}{2}(b - a + c - a)\lambda^2 + \ldots \pm M,$$

i.e. $x : y : z$ depend only on the differences of a, b, c, and the directions of the axes are the same for the two confocals.

(4) Let (f, g, h) be a point P on the ellipsoid $x^2/a + ... = 1$, and let the confocal hyperboloids be given by $x^2/(a-k) + ... = 1$, when g is very small, the two values of k are given by

$$(b-k)\{(c-k)f^2/a + (a-k)h^2/c\} + (a-k)(c-k)g^2/b = 0.$$

One value of k is, neglecting g^4,

$$b + g^2b^{-1}\{f^2/a(a-b) - h^2/c(b-c)\}^{-1},$$

and for the focal hyperbola $x^2/a(a-b) - z^2/c(b-c) = 0$; $\therefore$ the flat hyperboloid corresponding to this value of k will be of two or of one sheet as $f^2/a(a-b) >$ or $< h^2/c(b-c)$, in either case the normal at P will be nearly perpendicular to the plane zx; for the other hyperboloid $k = cf^2/a + ah^2/c$ nearly, the values of $a-k$, $b-k$, and $c-k$ being $(a-c)f^2/a$, $(b-c)f^2/a - (a-b)h^2/c$, $-(a-c)h^2/c$, nearly, the normals at P to it and the ellipsoid being close to the plane zx. If P be actually on the focal hyperbola, any line perpendicular to a tangent to the hyperbola will be a normal to either flat hyperboloid.

(5) $f^2/(a-b) - h^2/(b-c) = 1$ for the flat hyperboloid,

$$f^2/(a-b+k) + h^2/(c-b+k) = 1$$

for the ellipsoid, and by subtraction,

$$k = f^2(b-c)/(a-b) + h^2(a-b)/(b-c).$$

(6) Taking the axes and notation as in Art. 301, the vertex being on the focal hyperbola, if (l, m, n) be the direction of a side of the cone, r its length up to the point of contact,

$$r\{(p_1l + p_2m)/b - p_3n/k_3\} = 1,$$

and the tangent plane to the enveloping cone is $(xl+ym)/b - zn/k_3 = 0$, hence p, the perpendicular from the centre $(-p_1, -p_2, -p_3)$,

$$= \{(p_1l + p_2m)/b - p_3n/k_3\}\{(l^2+m^2)/b^2 + n^2/k_3^2\}^{-\frac{1}{2}};$$

$\therefore pr = \{(l^2+m^2)/b^2 + n^2/k_3^2\}^{-\frac{1}{2}}$, and $(l^2+m^2)/b = n^2/k_3 = 1/(b+k_3)$;

$$\therefore p^2r^2 = bk_3.$$

(7) $$\frac{l'}{l/b} = \frac{m'}{m/b} = \frac{n'}{-n/k_3} = pr = \frac{(a-b)(ll'+mm') + (a+k_3)nn'}{-1}.$$

(8) Let θ be the inclination of the cyclic plane of $x^2/a^2 + ... = 1$ to the plane xy; for the circular section $y^2 + x^2\sec^2\theta = b^2$, and $x^2/a^2 = \xi^2/(a^2-c^2)$, $y^2/b^2 = \eta^2/(b^2-c^2)$, $\sec^2\theta = b^2(a^2-c^2)/a^2(b^2-c^2)$;

$$\therefore \xi^2 + \eta^2 = b^2 - c^2.$$

(9) Let $x^2/a + y^2/b = 1$, $z = 0$ be the ellipse, treat it as a flat ellipsoid, the cone as an enveloping cone, whose vertex is (ξ, η, ζ), a point in the confocal $x^2/(a+k) + y^2/(b+k) + z^2/k = 1$, (1), the normal to which at the vertex is an axis of the cone, passing through the given point $(f, g, 0)$;

$\therefore (a+k)(f-\xi)/\xi = (b+k)(g-\eta)/\eta = -k$, $\therefore a(f-\xi)/\xi = -kf/\xi$, &c., and $a(f-\xi)/f = b(g-\eta)/g = -k = f\xi + g\eta - \xi^2 - \eta^2 - \zeta^2$, by (1), hence the locus is the section of a sphere by a plane.

XXVI.

(1) If $x^2+y^2+z^2=r^2$ be a maximum or minimum,

$$\sqrt{3}\,(z-y)/x=(z-x\sqrt{3})/y=(y+x\sqrt{3})/z=2/r^2=\frac{z+y}{y+z};$$

$$\therefore\ r^2=2,\ \text{or}\ -y=+z=x/r^2\sqrt{3}\ \text{and}\ 2z^2\sqrt{3}=x^2\sqrt{3}-xz,$$

whence $r^2=1$ or $-\frac{2}{3}$; the focal ellipse, referred to the axes, is $\frac{3}{8}x^2+\frac{3}{5}y^2=1$, and the focal hyperbola $x^2-\frac{3}{5}z^2=1$, eccentricities are $\sqrt{\frac{3}{8}}$ and $\sqrt{\frac{8}{3}}$.

Aliter, turn the axes of y and z through $-45°$, and then the axes of x and y through θ, where $\sin 2\theta/2\sqrt{6}=-\cos 2\theta=\frac{1}{5}$.

(2) $\quad \varpi^{-2}=\xi^2/(a+\lambda)^2+\ldots,\quad$ let $u\equiv\xi^2/(a+\lambda)^3+\ldots,$

$$0=\frac{2\xi}{a+\lambda}-\frac{d\lambda}{d\xi}\frac{1}{\varpi^2},\quad -\frac{d\varpi^2}{\varpi^4 d\xi}=\frac{2\xi}{(a+\lambda)^2}-\frac{4\xi\varpi^2}{a+\lambda}u,$$

$$\frac{d^2\lambda}{d\xi^2}=\frac{2\varpi^2}{a+\lambda}+\frac{d\varpi^2}{d\xi}\cdot\frac{2\xi}{a+\lambda}-\frac{4\xi^2\varpi^4}{(a+\lambda)^3}=\frac{2\varpi^2}{a+\lambda}-\frac{8\xi^2\varpi^4}{(a+\lambda)^3}+\frac{8\xi^2\varpi^6}{(a+\lambda)^2}u.$$

(3) For the circular section $x\sin\theta+z\cos\theta=pd/d_0$, where

$$\sin\theta:\cos\theta:1=c\beta:a\sqrt{(\gamma^2-\beta^2)}:b\gamma;$$

by Art. 286, $\quad \beta\gamma x=aa'a'',\quad \gamma\sqrt{(\gamma^2-\beta^2)}\,z=cc'c'';$

$$\therefore\ caa'a''+acc'c''=pb\gamma^2d/d_0,\quad\text{and}\quad pb=ac;$$

$$\therefore\ (a'^2-\gamma^2)(a''^2-\gamma^2)=(\gamma^2d/d_0-a'a'')^2;$$

$$\therefore\ \gamma^2(1-d^2/d_0^2)=a'^2+a''^2-2a'a''d/d_0.$$

(4) Art. 312. Let $(l,\ m,\ n)$ be the direction of a normal at $(\xi,\ \eta,\ \zeta)$ to the confocal $y^2/(b-k)+z^2/(c-k)=4(x-k)$, of which the focal conics are $y^2=4(b-c)(x-c)$ and $z^2=4(c-b)(x-b)$;

$$(b-k)m/\eta=(c-k)n/\zeta=-\tfrac{1}{2}l=(b-c)/(\eta/m-\zeta/n),$$

$$\therefore\ m\eta+n\zeta=-2l(\xi-k)=-2l\xi+l^2\eta/m+2lb,$$

$$\text{and}\ \eta/m-\zeta/n=-2(b-c)/l,$$

shew that, when $\eta=0$, $\zeta^2=4(c-b)(\xi-b)$; and similarly for the other focal conic.

(5) Use Art. 268, iii., and Art. 312. Let $(\xi,\ \eta,\ \zeta)$ be the point where a bifocal line of the paraboloid $y^2/b+z^2/c=4x$ meets a tangent plane, so that $l(l\xi+m\eta+n\zeta)+m^2b+n^2c=0$ (1); the bifocal line $(x-\xi)/l=r$, &c. intersects the focal conic,

$z=0,\ y^2=4(b-c)(x-c);\ \therefore\ (n\eta-m\zeta)^2=4(b-c)n\{n(\xi-c)-l\zeta\}$;

similarly, $\quad (n\eta-m\zeta)^2=-4(b-c)m\{m(\xi-b)-l\eta\}$;

$$\therefore\ n^2(\xi-c)+m^2(\xi-b)-l(m\eta+n\zeta)=0,\quad\text{or, by (1), }\xi=0.$$

(6) The extremities of ds are given by the intersection of the curve $x^2/a+\ldots=1$, $x^2/(a-k)+\ldots=1$ with the two hyperboloids

$$x^2/(a-k')+\ldots=1\ \text{and}\ x^2/(a-k'-dk')+\ldots=1;$$

but $(a-b)(a-c)x^2 = a(a-k)(a-k')$,

$$\therefore\ 4(a-b)(a-c)(dx/dk')^2 = a(a-k)/(a-k');$$

let (ξ, η, ζ) on the sphere correspond to (x, y, z) on the ellipsoid,

$$\therefore\ (a-b)(a-c)\xi^2 = r^2(a-k)(a-k'),$$

hence $4(a-b)(a-c)\{r^2(dx)^2 - k'(d\xi)^2\} = r^2(a-k)(dk')^2$.

(7) Let $(x-f)/l = (y-g)/m = (z-h)/n = \lambda$ be the equation of a bifocal chord through (f, g, h) in the paraboloid;

$$\therefore\ (g+m\lambda)^2/b + (h+m\lambda)^2/c = f + l\lambda,$$

and $(m^2/b + n^2/c)\lambda = l - 2(mg/b + nh/c)$. (1)

Since the chord intersects the conic, $z = 0$, $y^2/(b-c) = x - \frac{1}{4}c$,

$$\therefore\ (ng - mh)^2/(b-c) = n^2(f - \tfrac{1}{4}c) - lnh;$$

similarly $(ng-mh)^2/(b-c) = m^2(-f + \frac{1}{4}b) + lmg$;

$$\therefore\ (ng-mh)^2/bc = (m^2/b + n^2/c)f - \tfrac{1}{4}(m^2+n^2) - l(mg/b + nh/c),$$

whence $(mg/b + nh/c)^2 - l(mg/b + nh/c) + \frac{1}{4}l^2 = \frac{1}{4}$,

or, by (1), $(m^2/b + n^2/c)\lambda = 1$.

(8) Let one of the sections of the ellipsoid $x^2/a^2 + \ldots = 1$ stand on the line $(x - x_0)/l = (y - y_0)/m = r$, $(x_0, y_0, 0)$ being the centre of the section; $\therefore\ lx_0/a^2 + my_0/b^2 = 0$, and

$$(l^2/a^2 + m^2/b^2)r^2 = 1 - x_0^2/a^2 - y_0^2/b^2;$$

if α be the distance between the centre and focus $(\xi, \eta, 0)$, $\xi = x_0 + l\alpha$, $\eta = y_0 + m\alpha$, and $\alpha^2 = (\rho^2 - c^2)(1 - x_0^2/a^2 - y_0^2/b^2)$, where

$$\rho^{-2} = l^2/a^2 + m^2/b^2,\ \xi^2/a^2 + \eta^2/b^2 = x_0^2/a^2 + y_0^2/b^2 + \alpha^2/\rho^2,$$

and $l\xi/a^2 + m\eta/b^2 = \alpha/\rho^2$;

$$\therefore\ (\rho^2 - c^2)(1 - \xi^2/a^2 - \eta^2/b^2) = \alpha^2c^2/\rho^2 = c^2\rho^2(l\xi/a^2 + m\eta/b^2)^2.$$

The locus touches the principal section $\xi^2/a^2 + \eta^2/b^2 = 1$; shew that it meets the focal conic $\xi^2/(a^2-c^2) + \eta^2/(b^2-c^2) = 1$, where $\{(b^2-c^2)m\xi - (a^2-c^2)l\eta\}^2 = 0$.

9. Using the notation of the last problem, l, m is known by

$$\{l^2(a^2-c^2)/a^2 + m^2(b^2-c^2)/b^2\}(1 - \xi^2/a^2 - \eta^2/b^2) = c^2(l\xi/a^2 + m\eta/b^2)^2,$$

or by $\{l^2(a^2-c^2)/a^2 + m^2(b^2-c^2)/b^2\}\{\xi^2/(a^2-c^2) + \eta^2/(b^2-c^2) - 1\}$

$$= c^2\{(b^2-c^2)m\xi - (a^2-c^2)l\eta\}^2/\{(a^2-c^2)(b^2-c^2)a^2b^2\};$$

$$\therefore\ \xi^2/a^2 + \eta^2/b^2 < 1, \quad \text{and} \quad \xi^2/(a^2-c^2) + \eta^2/(b^2-c^2) > 1.$$

XXVII.

(1) By Art. 361, they are the enveloping cylinders, whose axes are parallel to the asymptotes of the focal hyperbola.

(2) By Art. 366, if (l, m, n) be the direction of the line of intersection, the equation of the two tangent planes will be

$$(l^2/a^2 + m^2/b^2 - n^2/c^2)\{x^2 + y^2 + z^2 - (1 + a^2/c^2)z^2 + (a^2/b^2 - 1)y^2\}$$
$$= a^2(lx/a^2 + my/b^2 - nz/c^2)^2,$$

intersecting the cyclic planes where

$$(l^2/a^2+\ldots)(x^2+y^2+z^2)=a^2(lx/a^2+my/b^2-nz/c^2)^2.$$

(3) Let $(x, y, 0)$ and $(x', 0, z')$ be the points S and S', then $x^2+y^2=b^2-c^2+x^2e^2/e'^2$, and $x'^2+z'^2=-(b^2-c^2)+x'^2e'^2/e^2$;

$$\therefore SS'^2=(x-x')^2+y^2+z'^2=(xe/e'-x'e'/e)^2.$$

For the directrices, by Art. 345, $\xi-\xi'=x/e'^2-x'/e^2 \propto SS'$.

(4) Let $(f, 0, h)$ be the point in the focal line,

$$x/\sqrt{(a^2-b^2)}=z/\sqrt{(b^2+c^2)},$$

and let $lx+my+nz=0$ be the equation of a tangent plane, where $a^2l^2+b^2m^2-c^2n^2=0$, $(x-f)/l=y/m=(z-h)/n$ will be the perpendicular from $(f, 0, h)$; $\therefore x(x-f)+y^2+z(z-h)=0$, (1) and $a^2(x-f)^2+b^2y^2-c^2(z-h)^2=0$, eliminating y,

$$\text{since } f^2/(a^2-b^2)=h^2/(b^2+c^2),$$

we have $(a^2-b^2)\{x-f-\frac{1}{2}b^2f/(a^2-b^2)\}^2=(b^2+c^2)\{z-h+\frac{1}{2}b^2h/(b^2+c^2)\}^2$;

$$\therefore f\{x-f-\tfrac{1}{2}b^2f/(a^2-b^2)\}=h\{z-h+\tfrac{1}{2}b^2h/(b^2+c^2)\}.$$

The negative sign is inadmissible, since it would give

$$f(x-f)+h(z-h)=0,$$

$$\text{or, with (1), } (x-f)^2+y^2+(z-h)^2=0.$$

(5) If $(\xi, \eta, 0)$ on the focal ellipse correspond to $(x, y, 0)$ on the ellipsoid, $\xi^2/(a^2-c^2)=x^2/a^2$, $\eta^2/(b^2-c^2)=y^2/b^2$; at the focus of the flat ellipsoid $\xi=\sqrt{(a^2-b^2)}$, $\therefore x=a\sqrt{(a^2-b^2)}/\sqrt{(a^2-c^2)}$.

(6) Let f, g, h be coordinates of P; at G $x=(1-c^2/a^2)f$, $y=(1-c^2/b^2)g$; at P' and G' $x=\sqrt{(1-c^2/a^2)}f$, $y=\sqrt{(1-c^2/b^2)}g$; hence $P'G'$ is parallel to Oz, $PG=P'G'$ by Ivory's theorem.

(7) The reciprocal locus is that of a point on the sphere, the sum of whose distances from two fixed points on the sphere is constant. Prove the last part by infinitesimals.

XXVIII.

(1) For a central cyclic section let $x=x'\cos\theta$, $z=x'\sin\theta$, where $b^2\cos^2\theta/(b^2-c^2)=a^2/(a^2-c^2)$, and for the cylinder, by Art. 345, $(a^2-c^2)x^2/a^4+(b^2-c^2)y^2/b^4=1$,

$$\therefore x'^2/a^2+y^2/b^2=b^2/(b^2-c^2);$$

if S be the focus $OS^2\cos^2\theta=a^2(a^2-b^2)/(a^2-c^2)$, for the parallel section touching at the umbilic, $OS\cos\theta=$ distance of umbilic from plane yz.

(2) By Art. 356, SR bisects $\angle QSQ'$, $\angle SPR=90°$.

(3) Let $(r\cos\phi, 0, r\sin\phi)$ be the vertex V on the focal hyperbola, α the radius vector in direction OV, β the semi-diameter of the section (a, c), conjugate to α,; a', b' semi-axes of the section

by the plane of contact, ρ the distance of its centre from O, then $\alpha^2+\beta^2=a^2+c^2$, $\rho r=\alpha^2$, and $a'^2/\beta^2=1-\rho^2/\alpha^2=b'^2/b^2=(a'^2-b'^2)/(\beta^2-b^2)$ and $a^{-2}\cos^2\phi+c^{-2}\sin^2\phi=\alpha^{-2}$, $(a^2-b^2)^{-1}\cos^2\phi-(b^2-c^2)^{-1}\sin^2\phi=r^{-2}$; the square of the distance required

$$=b'^4/(a'^2-b'^2)=b^4(1-\alpha^2/r^2)/(a^2+c^2-b^2-\alpha^2)$$
$$=b^4\{\cos^2\phi/a^2+\sin^2\phi/c^2-\cos^2\phi/(a^2-b^2)$$
$$+\sin^2\phi/(b^2-c^2)\}/\{(a^2-b^2)\sin^2\phi/c^2-(b^2-c^2)\cos^2\phi/a^2\},$$

reducing to $b^6/\{(a^2-b^2)(b^2-c^2)\}$.

(4) Let (x_1, y_1, z_1) (x_2, y_2, z_2) be extremities of diameters conjugate to the diameter of $x^2/a+y^2/b+z^2/c=1$, whose direction is (l, m, n). The two planes $xx_1/a+\ldots=0$ and $xx_2/a+\ldots=0$ are conjugate and perpendicular, $\therefore$ $x_1x_2/a+\ldots=0$ and $x_1x_2/a^2+\ldots=0$; $\therefore x_1x_2 : y_1y_2 : z_1z_2=a^2(b-c) : b^2(c-a) : c^2(a-b)$,

also $lx_1/a+\ldots=0$, and $lx_2/a+\ldots=0$, or $la(b-c)/x_1+\ldots=0$, eliminating x_1, the quadratic in $y_1 : z_1$ is to be satisfied by an infinite number of values; $\therefore$ m or $n=0$, if $m=0$, $l^2/(a-b)=n^2/(b-c)$.

(5) Let S, C be the focus and centre of the sphero-conic, SY perpendicular to the tangent at P, O the vertex of the cone $\angle COS=\gamma$, $\sin^2\gamma/(a^2-b^2)=\cos^2\gamma/(b^2+c^2)$; and let $\lambda x+\mu y+\nu z=0$ be the equation of OPY, where $\lambda^2a^2+\mu^2b^2-\nu^2c^2=0$,

$\mu(x\cos\gamma-z\sin\gamma)+(\nu\sin\gamma-\lambda\cos\gamma)y=0$ that of SOY;

$$\therefore (\mu x-\lambda y)^2\cos^2\gamma=(\mu z+\nu y)^2\sin^2\gamma,$$
$$(\mu x-\lambda y)^2+(\lambda x+\mu y)^2=(\lambda^2+\mu^2)(x^2+y^2),$$
$$\therefore (\mu x-\lambda y)^2=x^2+y^2-\nu^2r^2,\ (\mu z-\nu y)^2=y^2+z^2-\lambda^2r^2;$$
$$\therefore (x^2+y^2-\nu^2r^2)(b^2+c^2)=(y^2+z^2-\lambda^2r^2)(a^2-b^2),$$
$$(x^2+y^2)(b^2+c^2)-(y^2+z^2)(a^2-b^2)=\{\nu^2(b^2+c^2)-\lambda^2(a^2-b^2)\}r^2=b^2r^2;$$
$$\therefore (x^2+y^2+z^2)(b^2+c^2-a^2)=z^2(b^2+c^2)-x^2(a^2-b^2).$$

Aliter. By spherical trigonometry, let S' be the other focus. $S'P+SP=2\alpha$, $CY=\rho$, $SY=\rho'$, $\angle CSY=\theta$, and let $S'P$, SY produced intersect in T, then $S'T=2\alpha$,

$$\cos\rho=\cos\gamma\cos\rho'+\sin\gamma\sin\rho'\cos\theta,$$
$$\cos2\alpha=\cos2\gamma\cos2\rho'+\sin2\gamma\sin2\rho'\cos\theta,$$
$$\text{whence } \sin^2\alpha=\cos^2\gamma+\cos^2\rho'-2\cos\gamma\cos\rho'\cos\rho.$$

If (l, m, n) be the direction of OY, $\cos\rho=n$,

$$\cos\rho'=l\sin\gamma+n\cos\gamma,\ \cos\rho'-\cos\gamma\cos\rho=l\sin\gamma;$$
$$\therefore \sin^2\alpha=\cos^2\gamma(1-n^2)+l^2\sin^2\gamma;\ \therefore a^2=(b^2+c^2)(1-n^2)+(a^2-b^2)l^2,$$
$$\text{or } (b^2+c^2-a^2)(x^2+y^2+z^2)=(b^2+c^2)z^2-(a^2-b^2)x^2.$$

The equation of the two tangent planes through the line

$x/l=y/m=z/n$ is $(l^2/a^2+\ldots)(x^2/a^2+\ldots)=(lx/a^2+\ldots)^2$,

and if they be perpendicular, the sum of the coefficients of x^2, y^2,

and z^2 will be zero; $\therefore (l^2/a^2 \ldots)(a^{-2}+b^{-2}-c^{-2}) = l^2/a^4 + \ldots$, whence

$$(a^2-c^2)(x^2+y^2+z^2) = x^2(a^2-b^2) - z^2(b^2+c^2)$$

is the locus of the line, a cone having the same cyclic sections as the cone above.

(6) Take a section through S perpendicular to VS, S is the focus of the section, Art. 372. Let tangents at P, P' intersect in Q, and the third tangent intersect these in T, T', $\angle TST' = \frac{1}{2}\angle PSP'$; and if $\angle TST' = 90^0$, PSP' will be a straight line, and Q on the directrix.

Reciprocal Theorem. VP, VP' two fixed sides of a cone, and VQ any other side, intersect a circular section in p, p', q, $\angle pqp'$ is constant.

(7) Taking the section as in (6), PSP' being a chord through S, $VS/SP + VS/SP'$ is constant.

Reciprocal Theorem. Two tangent planes intersect in a line in a cyclic plane, shew that the sum of the cotangents of the angles which the cyclic plane makes with the two tangent planes is constant.

XXIX.

(1) 1. Turn xOy through 45°, $3z^2 - (x^2+y^2) + 2(x^2-y^2) = a^2$.

2. $s^3 - \frac{3}{4}s - \frac{1}{4} = 0$, $s = 1, -\frac{1}{2}, -\frac{1}{2}$, $x^2 - \frac{1}{2}(y^2+z^2) = a^2$.

3. $s = -1, 2 \pm \sqrt{3}$, $x^2/\frac{1}{2}(\sqrt{3}+1)^2 + y^2/\frac{1}{2}(\sqrt{3}-1)^2 - x^2 = 1$.

The method of Art. 415 gives $x = z = y/(\pm\sqrt{3}-1)$, for $s = 2 \pm \sqrt{3}$; it fails for $s = -1$, but $s(x^2+y^2+z^2) - x^2 - y^2 - z^2 - 2xy - 2yz - 4zx = -2\{y + \frac{1}{2}(x+z)\}^2 - \frac{3}{2}(x+z)^2 = 0$; $\therefore x + z = 0, y = 0$ are the equations of the axis corresponding to $s = -1$.

4. The equation may be written $(x+y+z)^2 - z^2 = a^2$, shewing that it is a hyperbolic cylinder; and $s = 0$ or $\pm\sqrt{3}+1$; corresponding to $s = 0$, $z = 0$, and $x + y = 0$, for $s = \pm\sqrt{3}+1$, $xs = ys = z(s+1)$; $\therefore x = y = z/(\pm\sqrt{3}-1)$.

5. As in Art. 427, $2(z+y+4)^2 - (2y - \frac{5}{2}x + 3)^2 + \frac{5}{4}(x-6)^2 = 52$, the three conjugate planes intersect in the centre $(6, 6, -10)$.

6. $s = -1, 1 \pm \sqrt{2}$; for the axis corresponding to s, by Art. 415, $xs = y(1+s) = z(1+s)$; as in 3, when $s = -1$, $x = 0$, $y + z = 0$, and when $s = \pm\sqrt{2}+1$, $x/\pm\sqrt{2} = y = z$. Centre $(0, -1, 1)$ is on the surface.

7. Or $\{x + \frac{1}{2}(y+z-7)\}^2 + \frac{3}{4}(y + \frac{5}{3}z - 7)^2 + \frac{2}{3}(z-3)^2 + d - 55 = 0$.

8. Or $14(y-x)^2 - (z+3y-4x)^2 = 1$; axis $x = y = z$.

(2) The equation is $(a^2+b^2+c^2)(x^2+y^2+z^2)-(ax+by+cz)^2=1$, Art. 58.

(3) $s^3-(a^2+b^2+c^2)s^2+4a^2b^2c^2=0$, one root negative, $s_1^{-1}+s_2^{-1}+s_3^{-1}=0$.

(4) $ac-b'^2=(aa'-b'c')b'/c'$, &c., $\therefore$ the equation becomes $(ax+b'z+c'y)^2+(aa'-b'c')(b'z+c'y)^2/b'c'+a(2a''x+\ldots+d)=0$, a paraboloid whose axis is $x=0$, $b'z+c'y=0$.

(5) The equation is $(a-1)x^2+(2y+3z+x)^2+2a''x+\ldots+d=0$.

i. If $b''=2\beta, c''=3\beta, (a-1)x^2+(2y+3z+x+\beta)^2+2(a''-\beta)x+d-\beta^2=0$, $a=1$, a parabolic cylinder; $a>1$, an elliptic cylinder, or line-cylinder, or impossible, as

$$(a''-\beta)^2/(a-1)+\beta^2-d>,\ =,\ \text{or}\ <0;$$

$a<1$, a hyperbolic cylinder, or two planes, as

$$\beta^2-d-(a''-\beta)^2/(1-a)\ \text{is finite or zero.}$$

ii. If $a''=\beta$, $b''=2\beta$, $c''=3\beta$, and $a=1$, $(2y+3z+x+\beta)^2=\beta^2-d$, representing two planes parallel, or coincident if $\beta^2=d$.

(6) For a generator of the opposite system, $y-a=\alpha x$, $z-a=\beta y$, where $-a=\beta(a+\alpha a)$; $\therefore yz+zx+xy-a(x+y+z)+a^2=0$ is the equation of the hyperboloid, centre $(\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a)$; referred to the centre and axes it is $x^2-\frac{1}{2}(y^2+z^2)+\frac{1}{4}a^2=0$, the eccentricity is $\sqrt{\frac{3}{2}}$.

(7) $s^3-\frac{7}{4}s+\frac{3}{4}=0$, $s=1, \frac{1}{2}, -\frac{3}{2}$; the corresponding direction-cosines of the axes are as $-\sqrt{3}:1:-1$, $0:1:1$, and $\frac{2}{3}\sqrt{3}:1:-1$, those corresponding to $s=\frac{1}{2}$ are obtained from the two factors of $u_2-\frac{1}{2}(x^2+y^2+z^2)$ equated to zero, the result of Art. 415 giving an indeterminate result in this case. The focal conics are $y^2-\frac{3}{5}z^2=1$ and $\frac{3}{5}x^2+\frac{3}{8}y^2=1$, eccentricities $\sqrt{\frac{8}{3}}$ and $\sqrt{\frac{3}{8}}$.

(8) For the centre, $x+pz-a=0$, $y+qz-b=0$, $-z+px+qy+c=0$.

i. $x(x-a)+y(y-b)+z(z-c)=0$, the locus is a sphere.

ii. When the centre is on the surface, $ax+by-cz=0$ and $x^2+y^2+z^2-2cz=0$, the locus is a circle.

(9) By Art. 414, either b' or $c'=0$; let $b'=0$, then $c'^2=(c-a)(c-b)$, the section by the plane of xy is a parabola whose axis is that of the paraboloid, $\therefore c'^2=ab$ and $c=a+b$; the equation of the section is $(x\sqrt{a}+y\sqrt{b})^2+2a''x=0$, and that of the diameter bisecting chords in direction (l, m) is $(l\sqrt{a}+m\sqrt{b})(x\sqrt{a}+y\sqrt{b})+a''l=0$, which will be that of the axis, if it cut the chords at right angles, *i.e.* if $l/\sqrt{a}=m/\sqrt{b}$; $\therefore$ the equations of the axis of the paraboloid are $(a+b)(x\sqrt{a}+y\sqrt{b})+a''\sqrt{a}=0$, $z=0$.

XXX.

(1) Treating the two planes as a conicoid, the bisecting planes are principal planes corresponding to the roots s_1, s_2 of the cubic, the third principal plane being any plane perpendicular to the line $x/A = y/B = z/C$, where $A = (aa' - b'c')^{-1}$, &c. Let (λ, μ, ν) be the direction of the normal to either of the two planes, then for any point in either, by Arts. 417, 418, the following equations hold $(ax + c'y + b'z)\lambda + \ldots = 0$, $\lambda x + \mu y + \nu z = 0$ and $\lambda A + \mu B + \nu C = 0$.

(2) Let λ, μ, ν and λ', μ', ν' be the minors of the discriminant of $ax^2 + \ldots + 2a'yz + \ldots - \alpha(x^2 + y^2 + z^2)$, α, β, γ being the roots of the discriminating cubic $f(s) = 0$. Since the discriminant vanishes, as in Art. 391, $\lambda\lambda' = \mu'\nu'$, &c., $\therefore \lambda\lambda'^2 = \mu\mu'^2 = \nu\nu'^2$, also $l\lambda' = m\mu' = n\nu'$, $\therefore l^2/\lambda = m^2/\mu = n^2/\nu = (\lambda + \mu + \nu)^{-1}$, and $\lambda + \mu + \nu = f'(\alpha) = (\alpha - \beta)(\alpha - \gamma)$.

(3) Let the equation be $\beta y^2 + \gamma z^2 + \ldots = 0$ by transformation, if θ be the angle required, $\tan\frac{1}{2}\theta = \sqrt{(\beta/-\gamma)}$;

$$\therefore \tan\theta = 2\sqrt{(-\beta\gamma)}/(\beta + \gamma) = 2\sqrt{(-I_2/I_1)},$$

with the notation of Art. 413.

(4) If the line $(\xi - x)/\lambda = \ldots = r$ meet the surface in two coincident points, $2u\left(\lambda^2\frac{d^2u}{dx^2} + \ldots + 2\mu\nu\frac{d^2u}{dy\,dz} + \ldots\right) = \left(\lambda\frac{du}{dx} + \ldots\right)^2$. Take three lines through the same point (x, y, z) at right angles, and add the corresponding equations;

$$\therefore 2u\left(\frac{d^2u}{dx^2} + \ldots\right) = \left(\frac{du}{dx}\right)^2 + \ldots.$$

(5) $s = 0$ or $a + b + c \pm \sqrt{\{(a + b + c)^2 - 3(bc + ca + ab)\}}$.

(6) Compare with $(x^2 + y^2 + z^2)\cos^2\frac{1}{2}\theta = (lx + my + nz)^2$, Art. 60.

(7) Let the plane be $lx + my + nz = p$, the equation of the cone is $p^2(ax^2 + by^2 + cz^2) - (lx + my + nz)^2 = 0$, which must be the same as $A(x^2 + y^2 + z^2) - B(\lambda x + \mu y + \nu z)^2 = 0$. Shew that l, m, or n must $= 0$, unless $a = b = c$, and that if $m = 0$, $\mu = 0$, and

$$p^2 = l^2/(a - b) + n^2/(c - b).$$

(8) Let the transformed equation be $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$, the given plane is $x\sqrt{\alpha} + y\sqrt{\beta} + z\sqrt{\gamma} = 1$, and, by Arts. 237, 240, the area of the section is $\pi\{3\alpha\beta\gamma/(\alpha + \beta + \gamma)\}^{-\frac{1}{2}}(1 - \frac{1}{3})$.

(9) A sphere, whose centre is in the axis, cuts the surface in two parallel planes; the left side of the equation must therefore be $A^2(x^2 + y^2 + z^2 + 2yz\cos\alpha + 2zx\cos\beta + 2xy\cos\gamma\} - \{A(x \pm y \pm z)\}^2$; $\therefore (\cos\alpha \pm 1)/a = $ &c.: the four forms in which $(x \pm y \pm z)^2$ can appear are $(x + y + z)^2$, $(x + y - z)^2$, &c.

(10) The equation of the cone referred to the vertex as origin is $\sigma(ax^2+by^2+cz^2)=(afx+bgy+chz)^2$, where σ stands for $af^2+bg^2+ch^2-1$; if s_1, s_2, s_3 be written for $s-\sigma a$, $s-\sigma b$, $s-\sigma c$, the discriminating cubic will assume the form

$$s_1s_2s_3+a^2f^2s_2s_3+b^2g^2s_3s_1+c^2h^2s_1s_2=0,$$

$$\text{or } a^2f^2/(s-\sigma a)+b^2g^2/(s-\sigma b)+c^2h^2/(s-\sigma c)+1=0,$$

also $af^2+bg^2+ch^2=1+\sigma$, multiplying the first by σ, and adding, we obtain $f^2/(a^{-1}-\sigma s^{-1})+g^2/(b^{-1}-\sigma s^{-1})+h^2/(c^{-1}-\sigma s^{-1})=1$. By Art. 415, the direction-cosines of the axis corresponding to s are inversely proportional to $a'(s-a+b'c'/a')$, &c., or to $bcgh(s-\sigma a)$, &c.; they are therefore as $f/(a^{-1}-\sigma s^{-1}):g/(b^{-1}-\sigma s^{-1}):h/(c^{-1}-\sigma s^{-1})$; hence the axis is a normal to a confocal passing through (f, g, h).

XXXI.

(1) The equation is $u_2=(2m-1)(m-1)$, and the cubic reduces to $s^3-(4m^2+3)s^2+4m^2(m^2+2)s+4(1-m^4)=0$. The surfaces are $m>1$ or <-1, $\alpha x^2+\beta y^2+\gamma z^2=+$, an ellipsoid; $m=1$, $4x^2+3y^2=0$, a line cylinder; $m<1>\frac{1}{2}$, $\alpha x^2+\beta y^2-\gamma z^2=-$, a hyperboloid of two sheets; $m=\frac{1}{2}$, $\alpha x^2+\beta y^2-\gamma z^2=0$, a cone; $m<\frac{1}{2}>-1$, $\alpha x^2+\beta y^2-\gamma z^2=+$, a hyperboloid of one sheet; $m=-1$, $4x^2+3y^2=6$, an elliptic cylinder.

(2) $x^2+y^2+z^2-r^2(ayz+bzx+cxy)=0$ gives two planes

$$(lx+my+nz)(x/l+y/m+z/n)=0,$$

equating coefficients

$$-abcr^6=(m^2+n^2)(n^2+l^2)(l^2+m^2)/l^2m^2n^2=(m/n+n/m)^2+\ldots-4.$$

(3) The equation of every surface of revolution through $x=0$, $y=0$ is of the form $n^2(x^2+y^2+z^2)-(lx+my+nz)^2+2Ax+2By=0$, and when it passes through $y=a$, $z=0$; $n^2-l^2=0$, $A=lma$ and $(n^2-m^2)a^2+2Ba=0$; hence the equation becomes

$$(l^2-m^2)(y^2-ay)-2lmx(y-a)\pm 2(lx+my)lz=0$$

$$\text{or } (l^2-m^2)(y^2-ay\pm xz)-2lm\{x(y-a)\mp yz\}=\mp(l^2+m^2)xz.$$

(4) $ax^2+by^2+cz^2$ must be of the form

$$\alpha(x^2+y^2+z^2+2yz\cos\lambda+2zx\cos\mu+2xy\cos\nu)-\beta(lx+my+nz)^2,$$

$$\therefore \alpha=a+\beta l^2=b+\beta m^2=c+\beta n^2=\beta mn/\cos\lambda=\beta nl/\cos\mu=\beta lm/\cos\nu$$

$$=\beta l^2\cos\lambda/\cos\mu\cos\nu=a\cos\lambda/(\cos\lambda-\cos\mu\cos\nu)\ \&c.$$

(5) $s(x^2+y^2+z^2)-ax^2-by^2-cz^2$ is transformed to

$$s(x^2+y^2+z^2+2a'yz+2b'zx+2c'xy)-2m(yz+zx+xy),$$

each, when equated to zero, being the equation of two planes;

$$\therefore s^3-\{(sa'-m)^2+(sb'-m)^2+(sc'-m)^2\}s+2(sa'-m)(sb'-m)(sc'-m)$$

$$\equiv(1-a'^2-b'^2-c'^2+2a'b'c')(s-a)(s-b)(s-c);$$

equating the coefficients gives the results.

If $a'=b'=c'=\frac{1}{2}$, $ab+bc+ca=0$ and $(a+b+c)^3=-\frac{27}{4}abc$, whence $s^3-(a+b+c)s^2+\frac{4}{27}(a+b+c)^3=0$, whence the three values of $s/(a+b+c)$ are $\frac{2}{3}$, $\frac{2}{3}$ and $-\frac{1}{3}$; $\therefore a=b=-2c$, &c.

(6) $s(x^2+\ldots+2a'yz+\ldots)-2(1-a')yz-2(1-b')zx-2(1-c')xy$ becomes a complete square if $s=-1$, and the transformed equation must be $ax^2-y^2-z^2=2d$, since $s(x^2+y^2+z^2)-u$ reduces to a complete square for $s=-1$, only when the coefficients of y^2 and z^2 are equal. The last term of the reducing cubic, being $a(-1)^2$, gives the result.

(7) Let the two discriminating cubics be $s^3-As-2B=0$ and $s'^3-A's'-2B'=0$, where $A=a^2+b^2+c^2$, $B=abc$, &c.: when the two conicoids are confocal $s'^{-1}=s^{-1}+k$,

$$\therefore s^3-A's(1+ks)^2-2B'(1+ks)^3=0,$$

and comparing the coefficients we have $B'/A'+B/A=0$ and $B'^2/A'^3+B^2/A^3=\frac{1}{27}$.

(8) Let $a-b'c'/a'=b-c'a'/b'=c-a'b'/c'=s$, then the equation of the surface of revolution is of the form

$$s(x^2+y^2+z^2)+a'b'c'(x/a'+y/b'+z/c')^2+2a''x+2b''y+2c''z+d=0,$$

and if (ξ, η, ζ) be the focus, $lx+my+nz-p=0$ the equation of the directrix plane, the equation is also

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2-(lx+my+nz-p)^2=0;$$

$$\therefore la'=mb'=nc'=(a'^{-2}+b'^{-2}+c'^{-2})^{-\frac{1}{2}}.$$

$$sl^2=-b'c'/a'=s-a \text{ \&c. } (1); \quad \therefore s=3s-a-b-c,$$

$$s(pl-\xi)=a'' \text{ \&c. } (2) \text{ and } s(\xi^2+\eta^2+\zeta^2-p^2)=d, \ (3)$$

$$\therefore \text{by } (2)\ a'(s\xi+a'')=b'(s\eta+b'')=c'(s\zeta+c'')=spla'$$

$$=\{s(a''\xi+b''\eta+c''\zeta)+a''^2+b''^2+c''^2\}/(a''/a'+b''/b'+c''/c'),$$

$$\text{and } s^2p^2=s^2(\xi^2+\eta^2+\zeta^2)+2s(a''\xi+b''\eta+c''\zeta)+a''^2+b''^2+c''^2;$$

$$\therefore \text{by } (3)\ s(a''\xi+b''\eta+c''\zeta)+a''^2+b''^2+c''^2=\tfrac{1}{2}(a''^2+b''^2+c''^2-sd).$$

The equation of the directrix is $sla'(lx+my+nz-p)=0$, and by (1) $sl^2a'^2=-a'b'c'$

$$\text{and } sla'p=\tfrac{1}{2}(a''^2+b''^2+c''^2-sd)/(a''/a'+b''/b'+c''/c').$$

(9) Let a sphere, centre O, of unit radius, intersect the axes in x, y, z, the axis of rotation in A, and the lines joining O to any point in its first and second position in P and P'; let (l, m, n), (l', m', n'), (λ, μ, ν) be the directions of OP, OP' and OA; $\angle P'Ax=\phi$, $\angle P'AP=\theta$, $AP'=AP=\alpha$, and take Π the pole of AP', then, by $\Delta P'Ax$, $l'=\lambda\cos\alpha+\sin\alpha\sin Ax\cos\phi$, and, by ΔPAx,

$$l=\lambda\cos\alpha+\sin\alpha\sin Ax\cos(\phi-\theta), \text{ and } \cos\Pi x=\sin Ax\sin\phi;$$

$\therefore l=\lambda\cos\alpha+(l'-\lambda\cos\alpha)\cos\theta+\sin\alpha\cos\Pi x\sin\theta$, $\cos\alpha=\lambda l'+\mu m'+\nu n'$, and, by Art. 24, $\cos\Pi x\sin\alpha=\mu n'-\nu m'$. lr, mr, nr and $l'r$, $m'r$, $n'r$ are the coordinates of P and P', r being the same for both.

Taking the conicoid $ax^2+\ldots+2a'yz+\ldots=1$, the equation should be unaltered when we write $-x+2\lambda(\lambda x+\mu y+\nu z)$ or $-x+2\lambda p$ for x, &c.;

$$\therefore -4p\{a\lambda x+\ldots+a'(\mu z+\nu y)+\ldots\}+4p^2(a\lambda^2+\ldots+2a'\mu\nu+\ldots)\equiv 0;$$

$$\therefore s(\lambda x+\mu y+\nu z)\equiv(a\lambda+c'\mu+b'\nu)x+\ldots+\ldots,$$

which gives the equation (1) of Art. 418.

XXXII.

(1) A plane through the vertex contains only two points of the curve besides the vertex, hence a straight line cuts the cone in two points only.

(2) If it could cut in more than $n-1$ points, a plane through it and any other point of the curve would cut the curve in more than n points.

(3) Let P be the point of crossing, Q, R points on the two branches near P, each conicoid of the cluster passes through P, Q, and R, and is touched by the plane PQR in its limiting position. If P' be a second point of crossing, a plane through PP' and any other point contains 5 points of the base, which is possible only when the base is two plane curves.

(4) A conicoid can be drawn through the point P common to the straight line and curve, six other points on the curve and two others on the straight line, so that the line and curve both lie entirely on the conicoid, Art. 447.

A plane through the line meets the curve in two points besides P, and the line joining the two is a generator.

(5) The tangent plane at P to each of the conicoids must contain Q and a point on the curve consecutive to P.

(6) Take A, B, C, D, E for the five points, and let AE intersect the plane BCD in e, an infinite number of conics pass through B, C, D and e, each being the base of a cone, vertex A, of which AB is a generating line, similarly an infinite number of cones with vertex B and a generating line BA pass through the five points; and each pair of cones gives a cubic curve through the points. If six points be given only one conic can be drawn in each case. No four of the five points can lie in one plane.

(7) By (4) one conicoid can be drawn containing the curve and the line joining an arbitrary point O with any point of the curve, the other generator through O contains two points on the cubic curve, the plane through the two generators containing three.

(8) A straight line L intersects the projection in three points only, since only three points of the curve lie in the plane containing L and O the origin of projection. Also, by (7), one line through O contains two points of the curve and their projections coincide in a point P, hence any line L which passes through P contains two coincident points of the projection.

XXXIII.

(1) The plane cuts the curve at P, Q and a third point R, a conicoid can be drawn through P, Q, R, four points of the curve, the given point S of the chord, and a point in RS, the curve will lie entirely in the conicoid, RS and PSQ will both be generators, and S the point of contact.

(2) By Art. 451, the common generating line counts for four points.

(3) The given conicoid (A) contains the five points, let L and L' be two generators of the same system; a second conicoid (B) passes through the five points and three on L, and (A, B) is the base of a cluster, which is the cubic curve C and the line L; a third (B') with (A) forms the base of a cluster (A, B'), which is the curve C' and line L'; four of the eight points common to A, B and B' lie on L and L', two on each, $\therefore$ only four would be common to C and C', unless they coincided; hence for each system of generators there is only one cubic curve through the five points on A.

(4) Let the cluster be denoted by the equation

$$ax^2 + \ldots + 2a'yz + \ldots + 2a''x + \ldots + d + \lambda(\alpha x^2 + \beta y^2 + \gamma z^2 - 1) = 0,$$

for any individual of the cluster the centre is given by

$$(a + \lambda\alpha)\,\xi + c'\eta + b'\zeta + a'' = 0,$$

$$c'\xi + (b + \lambda\beta)\,\eta + a'\zeta + b'' = 0,$$

$$\text{and } b'\xi + a'\eta + (c + \lambda\gamma)\,\zeta + c'' = 0;$$

and if the centre lies in the plane $Ax + By + Cz + D = 0$,

$$A\xi + B\eta + C\zeta + D = 0.$$

Eliminating ξ, η and ζ, the result is a cubic in λ. Hence a plane intersects the locus of the centres in three points only.

(5) The quartic curve lies on a cubic surface if that surface pass through 13 of its points; a conic on one conicoid cuts the curve in 4 points and lies on the cubic if 7 of its points are on the cubic, 4 of which may be of the 13; hence for each conic 3 more points are required, making in all $13 + 3 + 3 \equiv 19$, and so fixing the cubic surface.

(6) For the fixed line, let $x = f + lr$, $y = g + mr$, $z = h + nr$, for the normal to the conicoid $ax^2 + by^2 + cz^2 = 1$, the foot of which is (ξ, η, ζ), $f - \xi + lr - a\xi\rho = 0$, &c.; eliminating r and ρ, we have another conicoid on which the feet lie.

(7) If $u = 0$, $u' = 0$, give the base, the condition that $u + \lambda u' = 0$ may be the equation of a cone, gives, by Art. 396, four values of λ. Let $u'=0$ (1), be one of the four cones, P the polar plane of its vertex V with respect to $u = 0$, the cone enveloping $u = 0$ with vertex V has four generating lines common to it and (1), each of which has two coincident points, and since these points lie on both $u = 0$ and $u' = 0$, the four lines are tangents to the base, and the points of contact lie on P.

(8) Any plane through O, the origin of projection, contains four points of the base, therefore the projection of the base is cut by a straight line in four points.

One conicoid of the cluster passes through O, and the two generating lines through O each pass through two points of the base. Let one of the generating lines meet the plane of projection in P, then every line through P cuts the projection of the base in two coincident points.

XXXIV.

(1) $\lambda(lx^2 - my^2) + \mu(my^2 - nz^2) + \nu(nz^2 - rw^2) = 0$ is the equation of any conicoid passing through seven of the points and two arbitrary points.

(2) Let A, B and C, D be the points in which the given lines intersect the conicoid, refer the conicoid to the tetrahedron of which AB, CD are edges, its equation being

$$ayz + bzx + cxy + a'xw + b'yw + c'zw = 0 \quad (1).$$

The common tangent of the sections by planes through AB and CD must be their line of intersection given by $y = \alpha x$, $w = \beta z$; hence the roots of the equation

$$a\alpha xz + bzx + c\alpha x^2 + a'\beta xz + b'\alpha\beta xz + c'\beta z^2 = 0$$

must give equal values of $x : z$, or the equation is

$$\{2c\alpha x + (a\alpha + b + a'\beta + b'\alpha\beta)\, z\}^2 = 0,$$

and if $(\xi, \eta, \zeta, \omega)$ be the point of contact, eliminating α and β,

$$2c\xi\eta + a\eta\zeta + b\zeta\xi + a'\xi\omega + b'\eta\omega = 0, \quad \text{or} \quad c\xi\eta = c'\zeta\omega.$$

The equations of the locus are (1) and $cxy = c'zw$.

Near D, neglecting the squares of small quantities x, y, z, the tangent to the quartic curve at D has the equations $z = 0$ and $a'x + b'y + c'z = 0$ and intersects AB. Similarly for the points A, B and C.

(3) Take ABC the triangular section, OAB, OBC, OCA the three planes, the three conics in which intersect OA, OB, OC each in two points. One conicoid can be drawn through these six points, and one more point on each of the three conics; hence the conicoid passes through five points on each of the conics, and therefore contains them entirely.

(4) The hyperboloid S intersects the cubic C in the quartic curve Q and the two lines L, M of the same system, see Art. 458; S and C' intersect in Q', L' and M'.

i. Let L, L' be of the same system; S, C and C' have 18 common points, 6 on L, M and C', 6 on L', M' and C, $\therefore$ 6 on S, Q and Q'.

ii. Let L, L' be of opposite systems.

4 of the 18 points are intersections of L, M with L', M', 2 are on Q and L', M', 2 on Q' and L, M, $\therefore$ 10 are on Q, Q'.

(5) Let C_5 denote a curve of the 5th degree, S_3, S_2 surfaces of the third and second degree, L a line.

C_5, L lie on S_3, S_2; C'_5, L' on S_3, S'_2; S_3, S_2 and S'_2 have 12 common points.

i. Let L, L' not intersect, 2 points of S'_2 lie on L; 2 of S_2 on L'; $\therefore$ 8 on C_5 and C'_5.

ii. Let L, L' intersect, 3 lie on L, L'; $\therefore$ 9 on C_5, C'_5.

iii. Let L, L' coincide, corresponding to 5 points, Art. 451; $\therefore$ 7 on C_5, C'_5.

iv. L' a generator of S_2, 5 on L', 2 or 1 more on L, as L' and L are of opposite or the same system; 5 or 6 on C_5, C'_5.

v. L' a generator of S_2, L of S_2', 9 or 8 on L and L', as they do not or do intersect; $\therefore$ 3 or 4 on C_5, C'_5.

(6) The quartic curve is the intersection of the conicoid S_2 with a cubic surface S_3, O an arbitrary point on S_2, join O to any point P of the curve, and let OQ be a generator of S_2, the plane POQ contains P and three points on OQ and S_3, the projection is therefore a triple point, and the projection of the curve is a quartic curve, as in XXXIII. (8), which can have only one triple point.

(7) Let $u + \lambda v = 0$ give the cluster; when $\lambda = \lambda'$, let the conicoid pass through a third point in AB, which is then a generating line, ab is a generator of the opposite system to AB. As the plane turns round AB, ab generates the conicoid $u + \lambda' v = 0$. E is a point in the base, plane abE meets the conicoid (λ') in a generator EF of the same system as AB, so that EF is a fixed chord.

(8) The cubic surface must contain $6 \times 3 + 1 \equiv 19$ points. As in (5), complete intersection of S_3, S_2 is C_6, that of S_3, S_3' is C_3', C_6'.

i. C_3', S_2 give 6 points generally, $\therefore$ C_6, C_6' give $18 - 6 \equiv 12$.

ii. When $C_3' \equiv L'$, C_2', L' a generator of S_2, common to S_2, S_3, S_3', gives 6 points, Art. 451, C_2', S_2 4 points, C_6', C_6 8 points; C_2' a section of S_2, common to S_2, S_3, S_3', gives 10, Art. 452, L', S_2 gives 2, C_6', C_6 6 points.

iii. When $C_3' \equiv L'$, M', N', if of the same system, C_3', S_2 give 12 points as in i; if one be of opposite system C_3', S_2 give 13; $\therefore$ C_6', C_6 give 6 or 5 points. L' a generator of S_2 gives 6 points, M' and N' each 2 or 1 point more as they are of the same or opposite system to L', C_6', C_6 give 8, 7, or 6 points. L', M' both on S_2 give 12 if of the same system, N' 2 or 0 more, as of the same or opposite system, C_6', C_6 4 or 6 points; if L', M' be of opposite systems they give 11, and N' one more, $\therefore$ C_6', C_6 give 6 points.

L', M', N' all on S_2 and of same system give 18 points, C_6', C_6 do not intersect, if one be of opposite system, they give 16 points, and C_6', C_6 give 2 points.

(9) Let L, M, N be the cones, and let L, M and L, N intersect in plane curves, planes intersecting in the common chord CD; refer to $ABCD$ where A and B are vertices of M and N. For the vertex of L, $l\alpha = m\beta$, $\gamma = 0$, $\delta = 0$; let L, M intersect in $\alpha = k\beta$, L, N in $\beta = k'\alpha$. The equations of L, M, N are

$$(l\alpha - m\beta)(A\gamma + B\delta) + C\gamma\delta = 0, \quad (lk - m)\beta(A\gamma + B\delta) + C\gamma\delta = 0,$$
$$\text{and } (l - mk')\alpha(A\gamma + B\delta) + C\gamma\delta = 0,$$

$\therefore$ M and N intersect in the plane $(l - mk')\alpha = (lk - m)\beta$.

XXXV.

(1) The tangent plane at (ξ, η, ζ) is $x/\xi + y/\eta + z/\zeta = 3$, the volume $\propto \xi\eta\zeta$, $\therefore$ constant.

(2) Since at any point (ξ, η, ζ) of the surface $\eta\zeta - a(\eta + \zeta) = a\eta\zeta/\xi$, for the tangent plane $a\xi\eta\zeta(x/\xi^2 + y/\eta^2 + z/\zeta^2) = a(\eta\zeta + \zeta\xi + \xi\zeta)$, $\therefore$ the intercepts are ξ^2/a, η^2/a, ζ^2/a.

(3) Art. 525. The normal at P is the radius PC of the generating circle, whose plane is inclined at $\angle\,\theta$ to that of zx, draw PM perpendicular to the plane of xy, the projections of CM and PC on Ox are equal, $\therefore$ $\cos\alpha = \sin\gamma\cos\theta$, similarly $\cos\beta = \sin\gamma\sin\theta$; the coordinates of P are $(c + a\sin\gamma)\cos\alpha/\sin\gamma$, $(c + a\sin\gamma)\cos\beta/\sin\gamma$, and $a\cos\gamma$.

i. γ constant, locus is two circles parallel to xy.

ii. $\alpha = \beta$, $\theta = \frac{1}{4}\pi$, locus is two circles in a plane through Oz bisecting $\angle\,xOy$.

(4) Let l, m, n be the direction-cosines of the normal,

$$F'(x)+F'(a)\,da/dx=l\rho,\quad f'(x)+f'(a)\,da/dx=l\sigma,\ \&c.,$$

ρ and σ being the same for the three direction-cosines, eliminating da/dx, the ratios $l:m:n$ are found.

(5) For the projection of the normal at

$(a\cos\theta\sin\alpha,\ b\sin\theta\sin\alpha,\ c\cos\alpha)$, $ax/\cos\theta-by/\sin\theta=(a^2-b^2)\sin\alpha$,

find the envelope, θ being the parameter.

(6) In the question, for x^2+y^2+ax read x^2+y^2+az. The tangent cone at the origin is $a^2z^2=(c^2-a^2)(x^2+y^2)$, which becomes $x^2+y^2=0$ when $a=0$, and $z^2=0$ when $a=c$. Writing r^2 for x^2+y^2, and $a=c\sin\alpha$, the equation becomes

$$(r\mp\tfrac{1}{2}c\cos\alpha)^2=c\sin\alpha\,(c\cos^2\alpha/4\sin\alpha-z),$$

the surface is generated by revolution of two parabolas about the axis of z, $z=(c^2-a^2)/4a$ is the equation of a singular tangent plane.

(7) Let $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n=r$ be the equation of a tangent to the curve of intersection of the surface with the tangent plane $a\alpha^2x+\ldots=1$; since (α,β,γ) is a double point on the curve, three values of r must be zero,

$$\therefore\ a\alpha^2l+\ldots=0\ \text{ and }\ a\alpha l^2+b\beta m^2+c\gamma n^2=0,$$

and if (l_1, m_1, n_1) and (l_2, m_2, n_2) be the directions of the tangents at the double point,

$$l_1l_2:m_1m_2:n_1n_2=(b\beta^3+c\gamma^3)/a\alpha:(c\gamma^3+a\alpha^3)/b\beta:(a\alpha^3+b\beta^3)/c\gamma;$$

the condition of perpendicularity gives the result.

(8) When z is indefinitely large, bx^2 vanishes compared with the other terms, and the asymptotic surface is $(x+y)^2=az$.

(9) A straight line drawn through any point (x, y, z) in the direction (λ, μ, ν) meets the surface in two points at infinity if $(ax-1)\lambda+by\mu+cz\nu=0$ (1), and $a\lambda^2+b\mu^2+c\nu^2=0$; (1) gives the asymptotic plane containing all asymptotes parallel to a side of the cone $ax^2+by^2+cz^2=0$, and the asymptotic surface is the cone $a(x-a^{-1})^2+by^2+cz^2=0$, of which every asymptotic plane is a tangent plane.

(10) The directions of asymptotic lines through a point (x, y, z) are given by $\lambda x/a^2+\mu y/b^2-\nu z/c^2=0$ (1), and $\lambda^2/a^2+\mu^2/b^2-\nu^2/c^2=0$; but, if $x/a=y/b$ and $z=0$, $\lambda/a=-\mu/b=\pm\nu/c\sqrt{2}$, $\therefore$ all the asymptotes lie in the plane (1) or $x/a-y/b\mp\sqrt{2}\,z/c=0$.

XXXVI.

(1) A corresponding surface where $\xi = x\sqrt{a}$ &c. has the equation $(\xi^2+\eta^2+\zeta^2)^2 - 3(\xi^2+\eta^2) - \zeta^2 + \frac{1}{4} = 0$, and is generated by the revolution about the axis of z of the curve on plane zx,

$$(\xi^2+\zeta^2)^2 - 3\xi^2 - \zeta^2 + \tfrac{1}{4} = 0, \text{ or } (\xi^2+\zeta^2-\tfrac{3}{2})^2 = 2(1-\zeta^2).$$

There are two double points on the axis of z at $\zeta = \pm(2)^{-\frac{1}{2}}$, the tangents at which are inclined at $\pm\frac{1}{4}\pi$ to the axis; $\zeta = \pm 1$ gives two double tangents. The former generate on revolution two conical points, the latter two singular tangent planes.

(2) The shortest distance between two consecutive generators is perpendicular to both and therefore to the fixed plane, the tangent plane at a point in the line of striction contains the shortest distance at that point and the corresponding generator, hence the normal is parallel to the fixed plane.

If (ξ, η, ζ) be a point in the paraboloid at which the normal is parallel to either asymptotic plane $x/a \mp y/b = 0$, $\xi/a^2 : \eta/b^2 = a : \pm b$.

(3) Every straight line in the plane $x = 0$ meets the surface at two points at infinity. See Art. 519.

(4) The plane $y = mx$ intersects the surface in another plane $z - a + (z-b)m^2 = 0$. The section by the plane $y = \beta$ touches the line $z = b$ where $x^2 = 0$, and $z = a$ is an asymptote, the curve lying between the two lines $z = a$ and $z = b$. When z is infinite alone, $x^2 + y^2 = 0$.

(5) By turning the axes of x and y through 45° the equation assumes the simpler form $z(2x^2 \pm b^2) + 2axy = 0$. Both surfaces can be generated by a straight line moving parallel to yz and intersecting the axis of x; for an infinite value of x, the generating line is in the plane of xy; for if $z = -my$, $m = 0$ when $x = 0$ or ∞.

With the upper sign m has a maximum value $a/b\sqrt{2}$.

With the lower sign as x changes from 0 to ∞ the generator twists from the plane xy through 180°, crossing the plane of xz where $x = b/\sqrt{2}$.

(6) Write the equation $(u-c)^2 = 4v$, the equation of the tangent plane is $(\xi - x)\{u - c - 2(u-x)\} + \ldots = 0$,

or $\xi(2x-u-c) + \ldots = 2(x^2+y^2+z^2) - (u+c)u = u^2 - 4v - cu = cu - c^2$.

$\Sigma\{(2y-u-c)(2z-u-c)\} = (u-c)^2 - 4u(u+c) + 3(u+c)^2 = 4c^2$,

$(2x-u-c)(2y-u-c)(2z-u-c) = 8xyz + 2c(u^2-c^2) = 4c^2(u-c)$;

$\therefore$ the sum of the intercepts $= c$.

(7) Let the equations of a generator of the surface be $x = mz + a$, $y = nz + b$, where m, n, a, and b are functions of one parameter θ, the variation of which gives rise to the different positions of the

generator. Show, by taking planes through each of two consecutive generators parallel to the other, that their shortest distance is

$$(\Delta m \Delta b \sim \Delta n \Delta a)/\sqrt{\{(\Delta m)^2 + (\Delta n)^2 - (m\Delta n - n\Delta m)^2\}};$$

$\Delta m = dm + \frac{1}{2}d^2m + \frac{1}{6}d^3m + \ldots$; and similarly for Δn, Δa, and Δb; $\therefore\ \Delta m\Delta b - \Delta n\Delta a = dmdb - dnda + \frac{1}{2}(dmd^2b + dbd^2m - dnd^2a - dad^2n)$ + terms of the following order higher than the third; the denominator is of the first order, and by the data $dmdb - dnda = f(\theta)(\Delta\theta)^2$ is zero for all values of θ, $\therefore$ $dmd^2b + dbd^2m - dnd^2a - dad^2n = 0$, and the numerator is of the fourth or higher order, whence the theorem, which is due to Bouquet.

(8) Let P and Q be $(0, 0, c)$ and (lr, mr, c), where $lmc = a$; shew that the tangent plane at Q has for its equation

$$(z - c)\, lmr = c\,(l^2 - m^2)(mx - ly) \quad (1);$$

and at the surface

$$(z - c)\, xy = c\,\{lm\,(x^2 + y^2) - (l^2 + m^2)\, xy\} = c\,(lx - my)(mx - ly);$$

hence, where the tangent plane meets the surface

$$xy\,(l^2 - m^2) = lmr\,(lx - my),$$

a hyperbolic cylinder, the section by the plane (1) is the hyperbola; the tangent to the section at P lies in the plane $lx - my = 0$.

(9) Let $z + c = 0$ be the equation of the horizontal plane, (ξ, η, ζ) the luminous point, referred to the axes of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Putting $z = -c$ in the equation of the enveloping cone, vertex at the luminous point, we have for the shadow the equation

$$(\xi^2/a^2 + \eta^2/b^2 + \zeta^2/c^2 - 1)(x^2/a^2 + y^2/b^2) = (x\xi/a^2 + y\eta/b^2 - \zeta/c - 1)^2.$$

In order that the shadow may be circular, ξ or η must vanish; let $\xi = 0$, the equation becomes

$$(\eta^2/b^2 + \zeta^2/c^2 - 1)\, x^2/a^2 + (\zeta^2/c^2 - 1)\, y^2/b^2 + 2\,(\zeta/c + 1)\, y\eta/b^2 = (\zeta/c + 1)^2;$$

equating the coefficients of x^2 and y^2, $(\zeta^2/c^2 - 1)(a^2 - b^2) = \eta^2$.

$\eta = 0$ would give an ellipse within the ellipsoid. The square of the radius is $a^2(\zeta/c + 1)/(\zeta/c - 1)$, which is independent of b for a given height of the luminous point.

(10) The plane $lx + my + nz = 1$ touches the given conicoids if $l^2a + m^2b + n^2c = 1$ and $l^2a' + m^2b' + n^2c' = 1$;

$$\therefore\ l^2\,(a\cos^2\theta + a'\sin^2\theta) + \ldots = 1,$$

shewing that the plane touches the third conicoid.

XXXVII.

(1) The tangent plane to the torse must contain tangents to both circles, which must therefore intersect in Ox; let their equations be $x\cos\theta + y\sin\theta = a$ and $x\cos\phi + z\sin\phi = c$, so that

$a \sec\theta = c \sec\phi$ (1); for the plane containing both,

$$x - a\sec\theta + y\tan\theta + z\tan\phi = 0,$$

and, when $x = 0$, $\sin\theta\, y/a + \sin\phi\, z/c = 1$, which is a tangent to the conic $y^2/\beta + z^2/\gamma = 1$ (2), if $\beta\sin^2\theta/a^2 + \gamma\sin^2\phi/c^2 = 1$, or, by (1), $(\beta+\gamma)\sin^2\theta/a^2 + \gamma(a^2 - c^2)/a^2c^2 = 1$; hence for all values of θ, $\gamma = -\beta = a^2c^2/(a^2 - c^2)$ and the torse touches (2).

(2) Writing ρ^2 for $x^2 - z^2$, the equation reduces to

$$3y^2(\rho \mp 2a) = (\rho \mp a)^2(\pm 4a - \rho),$$

$\therefore$ every real section parallel to zx consists of rectangular hyperbolas, the extremities of the transverse axes lie in the curves,

$$3y^2(x \mp 2a) = (x \mp a)^2(4a \mp x),$$

which have conjugate points where $x = \pm a$ and $y = 0$, the corresponding hyperbola being a conjugate line in zx.

(3) The section of the asymptotic cone by the plane at infinity has a double point, the tangents at which are the inflexional tangents, and, by Art. 475, the surface generally touches that plane.

(4) By Art. 517, $\lambda^2 = \pm\mu\nu$, and the inflexional asymptotes are the intersection of the planes $\pm 2x\lambda - y\nu - z\mu = 0$, and conicoids $6\lambda^2x^2 - (y\nu + 2z\mu)^2 + 3z^2\mu^2 \mp 2a^2\lambda^2 = 0$, and where they intersect $(y\nu - z\mu)^2 \mp 4a^2\lambda^2 = 0$. Hence the hyperboloid of one sheet gives real and that of two sheets imaginary inflexional asymptotes.

(5) The equation of the polar plane of a point (f, g, h), being $fx/(a+k) + gy/(b+k) + hz/(c+k) = 1$, involves one parameter, therefore, by Art. 484, it envelopes a torse. The foot (ξ, η, ζ) of one of the six normals from (f, g, h) to the confocal $x^2/(a+k') + \ldots = 1$ is given by $(a+k')(\xi - f)/\xi = \ldots = \ldots = \rho$, $\therefore\ \xi(a+k'-\rho) = f(a+k')$ &c., and the tangent plane at (ξ, η, ζ) is $fx/(a+k'-\rho) + \ldots = 1$, which is one of the polar planes enveloping the torse.

(6) The equation of the tangent plane to one of the confocals $x^2/(a+k) + \ldots = 1$ is $lx + my + nz = \{l^2a + m^2b + n^2c + k(l^2 + m^2 + n^2)\}^{\frac{1}{2}}$, and the tangent plane to the torse is common to all the confocals, $\therefore\ l^2 + m^2 + n^2 = 0$; let $m = il\cos\phi$, $n = il\sin\phi$, a point in the edge of regression is the intersection of three consecutive tangent planes, and its coordinates satisfy

$$x + iy\cos\phi + iz\sin\phi = (a - b\cos^2\phi - c\sin^2\phi)^{\frac{1}{2}}$$

$$-iy\sin\phi + iz\cos\phi = (b - c)\sin\phi\cos\phi\,(a - b\cos^2\phi - c\sin^2\phi)^{-\frac{1}{2}},$$

and $$-iy\cos\phi - iz\sin\phi = (b-c)(\cos^2\phi - \sin^2\phi)(a - b\cos^2\phi - c\sin^2\phi)^{-\frac{1}{2}}$$
$$-(b-c)^2\sin^2\phi\cos^2\phi\,(a - b\cos^2\phi - c\sin^2\phi)^{-\frac{3}{2}}.$$

The coordinates of the projection on the plane yz are given by

$$-iy = (b-c)(a-b)\cos^3\phi\,(a - b\cos^2\phi - c\sin^2\phi)^{-\frac{3}{2}},$$

$$iz = (b-c)(a-c)\sin^3\phi\,(a - b\cos^2\phi - c\sin^2\phi)^{-\frac{3}{2}},$$

$\therefore \{y\sqrt{(b-a)}\}^{\frac{2}{3}} + \{z\sqrt{(c-a)}\}^{\frac{2}{3}} = (b-c)^{\frac{2}{3}}$, which is the evolute of the focal conic $y^2/(b-a) + z^2/(c-a) = 1$.

(7) The plane $x+y+z=2$ is a triple tangent plane containing the lines of intersection with the coordinate planes. Transferring the origin to the point $(1, 1, 1)$, the equation is $yz+zx+xy+xyz=0$, the new axes lie entirely in the surface, and are generating lines of the conical tangent $yz+zx+xy=0$, but not double lines on the surface. For the last part, if (x, y, z) be the point of contact,

$$l : m : n : p = yz-1 : zx-1 : xy-1 : 2(x+y+z-3),$$

but $(zx-1)(xy-1) = x(x+y+z-2) - x(y+z) + 1 = (x-1)^2$,

$$\therefore\ 4p^{-2}mn = (x-1)^2/(x+y+z-3)^2, \text{ \&c.},$$

$$\therefore\ (mn)^{\frac{1}{2}} + (nl)^{\frac{1}{2}} + (lm)^{\frac{1}{2}} = \tfrac{1}{2}p.$$

(8) Take O for the origin and the normal for the axis of x, the equation of the conicoid will be

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy + 2a''x = 0.$$

Let the equations of a chord in the plane of xy be $z=0$ and $\alpha x + \beta y = 1$, $\therefore$ at the extremities of the chord

$$ax^2 + by^2 + 2c'xy + 2a''x(\alpha x + \beta y) = 0,$$

which gives the tangents of the angles subtended at O by the segments, $\therefore a + 2a''\alpha = Cb$, where C is the given constant; similarly if $y=0$ and $\alpha'x + \gamma z = 1$ be the equations of a chord in zx,

$$a + 2a''\alpha' = Cc.$$

Shew, by turning the axes Oy, Oz through any angle, that $b+c$ is unaltered, $\therefore \alpha + \alpha'$ is constant.

(9) Take $x^2+y^2=a^2$ for the circle, and the axis of x in the position of the generating line when in the plane of the circle; for any point in the generating line, when it has revolved round the tangent through an angle θ,

$$x = (a + z\cot\theta)\cos 2\theta \text{ and } y = (a + z\cot\theta)\sin 2\theta \quad (1);$$

all the generators pass through Oz, the line in which the surface intersects itself; the tangent plane at any point $(0, 0, h)$ in Oz is inclined to the plane zx at an angle 2θ, where $h = -a\tan\theta$.

As in Art. 494, for the projections of the shortest distance which is perpendicular to consecutive generators (θ) and $(\theta + d\theta)$,

$$\left.\begin{aligned} \cos 2\theta\,\delta x + \sin 2\theta\,\delta y + \tan\theta\,\delta z &= 0 \\ \text{and } -\sin 2\theta\,\delta x + \cos 2\theta\,\delta y - \tfrac{1}{2}\sec^2\theta\,\delta z &= 0 \end{aligned}\right\}\text{..............}(2),$$

also, by (1), writing r for $a + z\cot\theta$,

$$\delta x = -2r\sin 2\theta\, d\theta + \delta r\cos 2\theta,\ \ \delta y = 2r\cos 2\theta\, d\theta + \delta r\sin 2\theta;$$

$$\therefore \text{ by (2) } \tan\theta\,\delta z + \delta r = 0,\ \ 2r d\theta - \tfrac{1}{2}\sec^2\theta\,\delta z = 0,$$

$$\text{and } \delta r = \cot\theta\,\delta z - z\,\text{cosec}^2\theta\, d\theta,\ \ \therefore\ \tan\theta\,\delta z = z\,d\theta,$$

$$\text{whence } 2r = z/\sin 2\theta, \text{ or } 2y = z.$$

(10) Prove, as in Art. 495, i, that

$$-a^{\frac{3}{2}}x\sin\alpha + b^{\frac{3}{2}}y\cos\alpha - (-c)^{\frac{3}{2}}z = 0,$$

also $a^{\frac{1}{2}}x = \cos\alpha + (-c)^{\frac{1}{2}}z\sin\alpha$, $b^{\frac{1}{2}}y = \sin\alpha - (-c)^{\frac{1}{2}}z\cos\alpha$, whence z; and the second result comes from z being a maximum.

(11) Q, the polar plane of P, (f, g, h), with respect to the confocal $x^2/(a+k)+\ldots=1$ (1), intersects the torse in the line whose equations are $xf/(a+k)+\ldots=1$ and $xf/(a+k)^2+\ldots=0$ (2), in which Q intersects the consecutive polar plane; let (x', y', z') and (x'', y'', z'') be two points R' and R'' in this line, the polar line of $R'R''$ with respect to (1) is the intersection of $xx'/(a+k)+\ldots=1$ and $xx''/(a+k)+\ldots=1$; or, since R' and R'' are points in (2), $(x-f)x'/(a+k)+\ldots=0$ and $(x-f)x''/(a+k)+\ldots=0$; if (l, m, n) be the direction of this polar line $l:m:n=(y'z''-z'y'')(a+k):\ldots:\ldots$, but by the second equation of (2),

$$y'z''-z'y'':\ldots:\ldots=f/(a+k)^2:\ldots:\ldots,$$

$$\therefore\ l:m:n=f/(a+k):g/(b+k):h/(c+k);$$

hence polar line of $R'R''$ is perpendicular to Q, and its equations are

$$(a+k)(x-f)/f=(b+k)(y-g)/g=(c+k)(z-h)/h,$$

$\therefore$ the quadric cone generated is $(b-c)f/(x-f)+\ldots=0$, which is satisfied by $(a+k_1)(x-f)/f=\ldots=\ldots$, the normal to any confocal through P, and obviously by lines through P parallel to the axes.

XXXVIII.

(1) Take for the conicoid $ax^2+by^2+cz^2=1$, the polar of O, (f, g, h), is $afx+bgy+chz=1$, that of P, (ξ, η, ζ), $a\xi x+b\eta y+c\zeta z=1$, by the condition of perpendicularity $a^2f\xi+b^2g\eta+c^2h\zeta=0$, a plane diametral to chords whose direction-cosines are proportional to af, bg, ch.

(2) Take AB, CD of the fundamental tetrahedron for the common generators, the equations of the two hyperboloids will be $ayz+bzx+cxw+dyw=0$ and $a'yz+\ldots=0$; a generator of the opposite system, whose equations are $y=\lambda x$, $w=\mu z$, will be common to the two hyperboloids, if

$$a\lambda+b+c\mu+d\lambda\mu=0 \text{ and } a'\lambda+b'+c'\mu+d'\lambda\mu=0;$$

there will therefore be two such, and the four intersections with AB and CD will be the four points of contact.

(3) Let A be the point in which the four lines intersect, and let a conicoid pass through the arbitrary points on three of the lines, and one of the points P on the fourth line, cutting it in a second point P'; the polar plane of A with respect to this conicoid cuts the fourth line in a point R, then AP, AR, and AP' are in harmonic progression, and P' is therefore not an arbitrary point.

Hence the only conicoids which satisfy the conditions are cones passing through the four lines, and, as particular cases, pairs of planes each containing two of the four lines.

(4) $lzx + myw = 0$ is the form of the equation of the conicoid referred to $ABCD$, tetrahedral coordinates. The polar plane of the centre of gravity $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is $lx + my + lz + mw = 0$, which meets AC and BD where $x + y + z + w = 0$.

(5) The equations of the tangent cones at A and B are

$$mzw + nwy + ryz = 0 \text{ and } lzw + nwx + rxz = 0,$$

$nw + rz = 0$ is the equation of a common tangent plane, $ly - mx = 0$ is that of the common plane section, containing the edge CD opposite to the common line AB. The six plane sections meet in the point $x/l = y/m = z/n = w/r$.

(6) For the centre $lz + mw = nz + rw = lx + ny = mx + ry$, whence $x/(r-n) = y/(l-m) = z/(r-m) = w/(l-n)$. If the centre be at an infinite distance $x + y + z + w = 0$. For the line joining the middle points $(\frac{1}{2}, \frac{1}{2}, 0, 0)$ and $(0, 0, \frac{1}{2}, \frac{1}{2})$, $x - \frac{1}{2} = y - \frac{1}{2} = -z = -w$, $\therefore$ $x = y$ and $z = w$, and the line lies in the surface when

$$l + m + n + r = 0.$$

(7) The equation of the tangent plane at (x', y', z', w') is

$$l(y'x + x'y) - m(w'z + z'w) = 0. \quad (1)$$

i. At the points $(0, y', 0, w')$ and $(x', 0, z', 0)$ in BD, AC, for the tangent planes $ly'x - mw'z = 0$ and $lx'y - mz'w = 0$, and the line of intersection is on the plane $x + y + z + w = 0$,

$$\therefore y'/m = -w'/l = 1/(m-l) = x'/m = -z'/l,$$

and $x' = y'$, $z' = w'$, or the centre is in the line bisecting AB and CD, see (6).

ii. At P and Q, $lx^2 = ly^2 = mz^2 = mw^2$; $\therefore$ by (1), for the tangent planes, $(x+y)\sqrt{l} \pm (z+w)\sqrt{m} = 0$, shewing that they are parallel.

(8) It should have been stated in the problem that a is the intersection of the tangent planes at B, C, D, and similarly for b, c, and d.

$$\begin{array}{lll}
\text{Tangent plane at } A, & ny + mz + l'w = 0, & (1) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; B, & nx + lz + m'w = 0, & (2) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; C, & mx + ly + n'w = 0, & (3) \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; D, & l'x + m'y + n'z = 0. & (4)
\end{array}$$

i. Each of the points a and b lies in the planes (3) and (4), and therefore, eliminating x, in the plane $(ll' - mm')y - mn'z + l'n'w = 0$, which contains A. Also where Aa, Bb intersect, B is in the same plane, unless a and b coincide, hence, except in this case, $ll' = mm'$, the same condition as that for the intersection of Cc, Dd.

When a and b coincide, (1), (2), (3), and (4) are simultaneous equations, the same as when the centre is on the surface; thus the surface is a cone, a, b, c, d in the vertex.

ii. When the surface is not a cone, $abcd$ is a tetrahedron and $ll' = mm' = nn'$.

(9) If the surface were ruled the tangent plane at D would intersect the surface in two real lines, $\therefore\ x/l + y/m + z/n = 0$ (1) and $lyz + mzx + nxy = 0$ should give real values of $x : y$, but the roots of $x^2/l^2 + y^2/m^2 + xy/lm = 0$ are impossible.

The centre is given by $s/l - x/l^2 = s/m - y/m^2 = \ldots$, where $s = x/l + y/m + z/n + w/r$, whence, if $x + y + z + w = 0$,

$$s(l + m + n + r)/(l^2 + m^2 + n^2 + r^2) = (4s - s)/(l + m + n + r).$$

The intersection of the tangent plane at D with ABC by (1) satisfies the equation $x/l + y/m + z/n + w/r = 0$; similarly for A, B, and C.

XXXIX.

(1) Let SA be a perpendicular from S, the origin of reciprocation, Art. 554, and let a be the point which corresponds to the plane of the circle. Join S to P any point in the circle, and draw ap perpendicular to SP, and aQ parallel to SP meeting AP in Q. A plane through ap perpendicular to SP corresponds to P and is a tangent plane to the cone reciprocal to the circle, and aQ is a generating line of the cone reciprocal to that cone; and since the locus of Q is a circle, the plane perpendicular to SA is a cyclic plane, and aS is therefore a focal line of the cone reciprocal to the circle.

(2) The trace of the reciprocal of the conic section on its plane is a circle.

(3) Take the section APB cutting the focal line VS of the cone in S. By XXVII. (4) the locus of the feet of the perpendiculars from S on the tangent planes is a circle; hence the reciprocal of the cone with respect to S is a circle in the plane corresponding to V, which is therefore parallel to APB. Since P is in the plane containing S, and the point corresponding to that plane is in VS at an infinite distance, the plane corresponding to P is a tangent plane to the circular cylinder whose generators are parallel to VS; and this cylinder cuts the plane in a circle, the reciprocal of APB with respect to S, which is therefore one of the foci.

(4) The reciprocal theorem is that if a conicoid circumscribe a tetrahedron $ABCD$, and the tangent planes at A and B intersect the opposite faces in two lines which lie in the same plane, and

therefore intersect, then the same will be true for the two lines corresponding to C and D.

$$ayz + bzx + cxy + dxw + eyw + fzw = 0$$

being the equation of the conicoid, where the tangent plane at A meets BCD, $cy + bz + dw = 0$ and $x = 0$, and for the line corresponding to B, $cx + az + ew = 0$ and $y = 0$; the two lines intersect if $b/a = d/e$, which is also the condition that the other two lines intersect.

(5) The polar of (ξ, η, ζ) with respect to the auxiliary conicoid will be a tangent plane to the surface $ax^2 + \ldots = 1$, if its equation $(b'\zeta + c'\eta)\,x + \ldots = 1$ coincide with $ax'x + \ldots = 1$, where $ax'^2 + \ldots = 1$; $\therefore\ b'\zeta + c'\eta = ax'$, &c.

(6) Let ρ be the distance from the origin of the point (ξ, η, ζ) corresponding to the tangent plane $lx + my + nz = p$,

$$\therefore\ \xi x + \eta y + \zeta z = \rho p = ac\,;$$

but $(yz - 2ax)/\xi = (xz - 2ay)/\eta$, $\therefore\ \xi x + \eta y = 0$ and $\zeta z = ac$,

$$\therefore\ z\xi\eta + a(\xi^2 + \eta^2) = 0, \text{ or } c\xi\eta + \zeta(\xi^2 + \eta^2) = 0.$$

(7) Let (f, g, h) be the point O, (l, m, n) the direction of one of the lines OP, and let $ax^2 + by^2 + cz^2 = 1$ be the given conicoid. The equations of the reciprocal of OP are

$$afx + \ldots = 1 \text{ and } alx + \ldots = 0 \quad (1).$$

Show that the condition of perpendicularity is

$$f'/l + g'/m + h'/n = 0 \quad (2), \text{ where } f'/f = b^{-1} - c^{-1} \text{ \&c.} \quad (3);$$

hence the lines OP lie on the cone $f'/(x - f) + \ldots = 0$.

The envelope of the lines (1), subject to the condition (2), has equations $\sqrt{(f'ax)} + \sqrt{(g'by)} + \sqrt{(h'cz)} = 0$ and $afx + \ldots = 1$; if this be a parabola, its projection on the plane xy will be so also, and the condition is $f'/f + g'/g + h'/h = 0$, which is true by (3).

(8) Let a plane be drawn perpendicular to the common focal line through any point S in it, S will be a common focus of the three sections of the cones, and the reciprocals with respect to S in the plane of the sections are three circles, whose radical axes meet in a point, therefore the intersections of the common pairs of tangents which correspond to these lie in a straight line; hence, since the common tangent planes of the cones contain these tangents, the theorem follows.

(9) $(\xi, \eta, \zeta, \omega)$ being the pole of the tangent plane at (x', y', z', w'), $\xi x + \eta y - \zeta z - \omega w = 0$ is identical with

$$y'x + x'y - kw'z - kz'w = 0, \text{ and } x'y' = kz'w'.$$

XL.

(1) Since the minor axes are equal, the products of the perpendiculars on parallel tangent planes from the common focus S are equal in the two surfaces; also the powers of the two spheres which are the reciprocals of the surfaces with respect to S are equal, therefore S lies in the plane of the circle in which the spheres intersect. The reciprocal of this circle is the cylinder enveloping each of the two surfaces, and S is the focus of the section of the cylinder by the plane of the circle.

(2) Let E denote the conicoid, E' its reciprocal with reference to S; then there correspond, (i) to A, B, C in E, three tangent planes to E' at right angles, (ii) to the envelope of the plane ABC, the locus of the intersection of perpendicular tangent planes, which is a sphere concentric with E', see prob. XXII. (8), (iii) to S and its polar plane with respect to E, a plane at infinity and its pole, the centre O. Reciprocating back, the envelope of ABC is the reciprocal of the sphere, viz. a spheroid, whose focus is S and directrix plane the plane corresponding to O, which by (iii) is the polar plane of S with respect to E.

(3) Reciprocating with O for the origin of reciprocation, we have for V, a plane; for the cone, a conic in that plane; for the tangent planes, points on the conic; for VP, VQ and VR, chords of the conic; for P, Q and R, planes through those chords mutually at right angles; for the envelope of PQR, the locus of a point from which three perpendicular lines pass through the perimeter of the conic.

If the equations of the conic be $ax^2+by^2=1$, $z=0$, that of the locus will be, by XVI. (12), $ax^2+by^2+(a+b)z^2=1$; the envelope of PQR is the reciprocal conicoid.

(4) The tangent plane at (x', y', z', w') has the equation

$$l(my'+nz'+rw')x+\ldots=0,$$

which must be the same as that of the polar $a\xi x+\ldots=0$ of $(\xi, \eta, \zeta, \omega)$, the point in the reciprocal corresponding to the tangent plane; $\therefore\ l(my'+nz'+rw')/a\xi=\ldots$; for $a\xi/l+b\eta/m+\ldots$ write S, show that $lx' : my' : nz' : rw' = S-3a\xi/l : S-3b\eta/m : \ldots$, and that $a\xi x'+\ldots=0$, $\therefore\ S^2-3\{(a\xi/l)^2+(b\eta/m)^2+\ldots\}=0$.

(5) See Art. 525. Let OY be perpendicular on a tangent to the generating circle in any position, and take Q in OY such that $OQ.OY=R^2$; then, if θ be the inclination of OY to the plane of xy, $OY=a+c\cos\theta$, $OQ^2=r^2+z^2$, and $OQ\cos\theta=r$,

$$\therefore\ a^2(r^2+z^2)=(R^2-cr)^2 \quad (1) \text{ and } r^2=x^2+y^2.$$

The reciprocal surface is generated by the revolution of the

hyperbola (1) about Oz, which it intersects where $z=R^2/a$, near which point let $z=R^2/a+\zeta$; substituting in (1) and neglecting ζ^2 and r^2, $cr+a\zeta=0$; hence the angle of the conical tangent at the multiple point is $2\tan^{-1}(a/c)=\pi-2\tan^{-1}(c/a)$.

(6) Let (ξ, η, ζ) be the pole of the tangent plane at (x', y', z') with respect to the point (α, β, γ), $\therefore (\xi-\alpha)(x-\alpha)+\ldots=R^2$ and $(ax'+c'y'+b'z')x+\ldots=1$ are identical equations, hence

$$\xi-\alpha=(ax'+c'y'+b'z')\rho \text{ \&c. } (1),$$

$$\text{and } R^2+\alpha(\xi-\alpha)+\ldots=\rho \ (2), \quad \therefore x'(\xi-\alpha)+\ldots=\rho \ (3).$$

Eliminating y' and z' from equations (1),

$$\rho\Delta x'=A(\xi-\alpha)+C'(\eta-\beta)+B'(\zeta-\gamma) \text{ \&c., Art. 391;}$$

the result follows from (2) and (3).

(7) If (α, β, γ) be a point on a focal conic, the reciprocal surface of the last problem will be a surface of revolution, Art. 564, hence, by Art. 414,

$$\Delta\alpha^2-A-(\Delta\gamma\alpha-B')(\Delta\alpha\beta-C')/(\Delta\beta\gamma-A')=\ldots=\ldots \quad (1),$$

$$\text{and } B'C'-AA'=a'\Delta,$$

$$\therefore \{A\beta\gamma+\alpha(A'\alpha-B'\beta-C'\gamma)+a'\}/(\Delta\beta\gamma-A')=\ldots=\ldots.$$

When the equations (1) become indeterminate in particular cases, as when the surface $x^2/a+y^2/b+z^2/c=1$ is given, the second form of the condition for a surface of revolution in Art. 414 must be used.

(8) When the reciprocal of prob. (6) is a sphere,
$\Delta\alpha^2-A=\Delta\beta^2-B=\Delta\gamma^2-C$, and $\Delta\beta\gamma=A'$, $\Delta\gamma\alpha=B'$, $\Delta\alpha\beta=C'$;
$\therefore \Delta\alpha^2-A=B'C'/A'-A=\Delta a'/A'$, so that $A'/a'=B'/b'=C'/c'$, which is the condition that the original conicoid be one of revolution,

$$\text{and } (\Delta\alpha^2-A+B)(\Delta\alpha^2-A+C)=\Delta^2\beta^2\gamma^2=A'^2.$$

XLI.

(1) Since the vertex of any cone passing through the intersection of the two conicoids must lie in the common generator, the fundamental tetrahedron required in the article does not exist.

(2) $u+v^2=0$ and $u+v'^2=0$ are the forms of the equations of the conicoids touching that for which $u=0$; and they intersect where $v=\pm v'$.

(3) The equations must be of the forms $u+vw=0$, $u+vw'=0$, $u+vw''=0$; the conicoids intersect where $v=0$ and $w=w'=w''$.

(4) Referring the conicoids to the tetrahedron $ABCD$, their equations are $lx^2+my^2+nz^2+rw^2=0$ and $l'x^2+m'y^2+n'z^2+r'w^2=0$, and that of the cone whose vertex is A is

$$(lm'-l'm)y^2+(ln'-l'n)z^2+(lr'-l'r)w^2=0.$$

The tetrahedral coordinates of points in the plane BCD are proportional to the triangular coordinates in that plane referred to the triangle BCD.

(5) Refer to the tetrahedron whose angular points are the vertices of the cones determined by the eight points, the equations of the conicoids being $lx^2+my^2+\ldots=0$ and $l'x^2+m'y^2+\ldots=0$; $\lambda:\mu$ will determine a paraboloid if $(l\lambda+l'\mu)^{-1}+(m\lambda+m'\mu)^{-1}+\ldots=0$, giving three values.

(6) Let $\lambda u+\mu v=0$ be one of the cluster, (f, g, h, k), (f', g', h', k') two points P, Q in the fixed line; let $U=0$, $V=0$ be the polars of P with respect to $u=0$, $v=0$, and let $U'=0$, $V'=0$ be the polars of Q; the equations of the polar with respect to $\lambda u+\mu v=0$ are $\lambda U+\mu V=0$ and $\lambda U'+\mu V'=0$, $\therefore$ $UV'=VU'$, and U, U', &c. are linear functions.

(7) Using the form of the general equation of a sphere in Art. 588, since the sphere touches AB, $z=0$ and $w=0$,

$$\therefore\ (px+qy)(x+y)-c^2xy=0$$

gives equal values of $x:y$, $\therefore$ $4pq=(p+q-c^2)^2$, whence $c=p^{\frac{1}{2}}+q^{\frac{1}{2}}$, similarly $c'=r^{\frac{1}{2}}+s^{\frac{1}{2}}$.

Geometrically. The sphere touches the edges internally, or three externally and three internally; the three tangents from each angular points are equal, the result follows.

(8) Each of the non-intersecting lines containing three points must lie entirely in the surface, which cannot, therefore, be a cone; let AB, CD be these generating lines of the surface, P, Q the other two points; a plane containing AB and P meets CD in some point D, and DP meets AB in some point B, $\therefore$ DB containing three points is a generator, similarly AC containing Q is a generator.

Referring to the tetrahedron $ABCD$, the equations of any two surfaces containing the eight points are

$$lyz+nxw=0 \text{ and } l'yz+n'xw=0,$$

and those of the polars of $(\xi, \eta, \zeta, \omega)$ are

$$l(\zeta y+\eta z)+n(\omega x+\xi w)=0 \text{ and } l'(\zeta y+\eta z)+n'(\omega x+\xi w)=0,$$

and these will be fixed if either $\zeta=0$, $\eta=0$, or $\omega=0$, $\xi=0$, that is, when $(\xi, \eta, \zeta, \omega)$ lies in AD or BC.

(9) The equation of a conicoid containing AB and CD is $lyz+mzx+nxw+ryw=0$, and if the plane $Ax+By+Cz+Dw=0$ is a tangent plane at the point $(\xi, \eta, \zeta, \omega)$, its equation must be the same as $(m\zeta+n\omega)x+(l\zeta+r\omega)y+\ldots=0$; hence prove that

$$\xi:\eta:\zeta:\omega=Cr-Dl:Dm-Cn:Ar-Bn:Bm-Al,$$

$$\text{and } A\xi+B\eta+C\zeta+D\omega=0,$$

shewing that the condition of touching a plane is a linear equation between l, m, n, and r.

XLII.

(1) Using the equations of XLI. (4); for the centre of

$$(l\lambda+l'\mu)x^2+\ldots=0,\ l\lambda+l'\mu=\rho/x,\ m\lambda+m'\mu=\rho/y,\ \&c.,$$

and the four cones are found by eliminating λ and μ from any three of the four equations.

(2) As in XLI. (8) AB, CD of the fundamental tetrahedron may be taken as generating lines containing six of the points, and AC as a third generator containing the seventh point, the equation of the conicoid being of the form $lyz+nxw+ryw=0$; for a paraboloid the condition is $l+n=r$, and any two paraboloids intersect where $y(z+w)=0$, $w(x+y)=0$, giving the three generators and a fourth fixed line $z+w=0$, $x+y=0$ at infinity.

(3) Using tetrahedral coordinates, the equation of the conicoid is $mzx+nxw+ryw=0$, and the first bisecting plane is $x-y-z+w=0$; the pole being $(\xi, \eta, \zeta, \omega)$, this must be the same as

$$(m\zeta+n\omega)x+r\omega y+m\xi z+(n\xi+r\eta)w=0,$$

$$\text{whence } \xi:\eta:\zeta:\omega=-r:m+n:n+r:-m,$$

and the pole lies in the second bisecting plane $x-y+z-w=0$.

(4) Let $M=0$ be the equation of the tangent plane at an umbilic U of a conicoid, the equation of the conicoid will be $S-M^2=0$, where S is the equation of a sphere, $S-M^2=L^2$ will be that of a conicoid touching the former along the section by $L=0$; where $M=0$, $S=L^2$, and if P be any point of the section, PM perpendicular to the intersection of $M=0$ and $L=0$, $L\propto PM$ and $S=PU^2$, $\therefore PU\propto PM$.

(5) When $w=0$, the equation must be the equation of two planes, and as in Art. 91 the points of intersection satisfy the equations $c'x+by+a'z=0$ and $b'x+a'y+cz=0$, from which the result follows.

(6) Let three conicoids of the cluster be

$$u=ax^2+\ldots+2fyz+\ldots+2pxw+\ldots=0,$$

$$v=a'x^2+\ldots=0, \qquad w=a''x^2+\ldots=0,$$

and let $Ax+By+Cz+Dw=0$ be the polar of (x', y', z', w'), the same for each conicoid of the cluster,

$$\therefore (ax'+hy'+gz'+pw')/A=\&c.$$

and two similar equations, which 9 equations cannot generally be satisfied by the 6 ratios $x':y':z':w'$ and $A:B:C:D$.

If $(0, 0, 0, 1)$ and $w=0$ be the pole and polar which are fixed for all the conicoids, p, q, and r vanish for each of the three conicoids, and the intersections of the conicoids lie in two quadric

cones given by $(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy)/d = \ldots = \ldots$, whose vertex is at D, so that the seven points must lie on the four lines of intersection of these two cones.

(7) Let α, β, γ, δ be the lengths of the tangents to the sphere from A, B, C, D; the tetrahedral coordinates of the points of contact with AB and CD are

$$\beta/(\alpha+\beta),\ \alpha/(\alpha+\beta),\ 0,\ 0 \text{ and } 0,\ 0,\ \delta/(\gamma+\delta),\ \gamma/(\gamma+\delta),$$

and $\alpha x = \beta y$, $\gamma z = \delta w$ are planes containing both; $\therefore$ $\alpha x = \beta y = \gamma z = \delta w$ is a point which lies in the line joining the points of contact of AB and CD, similarly for the other opposite edges.

(8) The polar of the centre $(\xi,\ \eta,\ \zeta,\ \omega)$ is at an infinite distance, so that $l\xi = m\eta = n\zeta = r\omega$, and the centre of a paraboloid being at an infinite distance $\xi + \eta + \zeta + \omega = 0$. The paraboloid will be hyperbolic or elliptic as the equations $lx^2 + \ldots = 0$ and $x+y+z+w=0$ give a real or imaginary curve, shew from

$$lx^2 + my^2 + nz^2 + r(x+y+z)^2 = 0,$$

$$\text{that } \{(l+r)x + r(y+z)\}^2 = lmnr(y/n - z/m)^2.$$

The equation of $b'c'a$ is $x = \frac{1}{2}$ or $-x + y + z + w = 0$, which is a tangent plane at a point whose coordinates are proportional to $-l^{-1}$, m^{-1}, n^{-1}, r^{-1}. Similarly for the planes $c'a'b$, $a'b'c$ and abc omitted in the statement. The equation of the plane $bcb'c'$ is $x - y - z + w = 0$, also a tangent plane, the point of contact being given by l^{-1}, $-m^{-1}$, $-n^{-1}$, r^{-1}. The plane in which the three points of contact with $c'a'b$, $a'b'c$, and abc lie, has the equation

$$-lx + my + nz + rw = 0.$$

The four lines mentioned in the problem all lie in the plane $lx + my + nz + rw = 0$.

(9) Let ABC of the fundamental tetrahedron be the plane containing five points and therefore a fixed conic, $w = 0$ and $lx^2 + my^2 + nz^2 = 0$; let D be one of the two points, so that the equation of the conicoid will be

$$lx^2 + my^2 + nz^2 + 2w(\lambda x + \mu y + \nu z) = 0 \quad (1);$$

the second point gives a linear equation between λ, μ, and ν.

Let $Ax + By + Cz + Dw = 0$ be the equation of a tangent plane to (1) at (x', y', z', w'), so that

$$lx' + \lambda w' = \rho A,\quad my' + \mu w' = \rho B,\quad nz' + \nu w' = \rho C,$$

$$\lambda x' + \mu y' + \nu z' = \rho D, \text{ and } Ax' + By' + Cz' + Dw' = 0;$$

find x', y', and z' from the first three equations, and by the last two shew that $\rho(A\lambda/l + \ldots - D) = w'(\lambda^2/l + \ldots)$

$$\text{and } \rho(A^2/l + \ldots) = w'(A\lambda/l + \ldots - D),$$

giving the quadratic equation

$$(A\lambda/l + \ldots - D)^2 = (A^2/l + \ldots)(\lambda^2/l + \ldots).$$

Each of the given tangent planes gives a quadratic equation which, combined with the linear equation above, supplies four systems of values of λ, μ, ν.

XLIII.

(1) This property holds for any four spheres, Art. 137.

(2) The equation of the circumscribing sphere is, Art. 587,

$$a^2yz + b^2zx + c^2xy + a'^2xw + b'^2yw + c'^2zw = 0 \quad (1).$$

The centre O' is given, Art. 589, by the equations

$$\begin{array}{llll} & c^2y + b^2z + a'^2w & = 2R'^2, \\ c^2x & \quad\;\; + a^2z + b'^2w & = 2R'^2, \\ b^2x + a^2y & \qquad\qquad + c'^2w & = 2R'^2, \\ a'^2x + b'^2y + c'^2z & & = 2R'^2. \end{array}$$

The square of the distance from O' to any point P, $(\xi, \eta, \zeta, \omega)$,

$$= - a^2(y-\eta)(z-\zeta) - b^2(z-\zeta)(x-\xi) - c^2(x-\xi)(y-\eta)$$
$$- a'^2(x-\xi)(w-\omega) - \dots .$$

The coefficient of ξ is $c^2y + b^2z + a'^2w, \equiv 2R'^2$,

$$\therefore\ O'P^2 = - R'^2 + 2R'^2(\xi+\eta+\zeta+\omega) - a^2\eta\zeta - b^2\zeta\xi - \dots .$$

Thus, if P be O, $(R/p_0, R/q_0, \dots)$ the centre of the inscribed sphere, $O'O^2 = R'^2 - R^2(a^2/q_0r_0 + \dots)$.

(3) Taking the four points as the angles of the fundamental tetrahedron, the equation is of the form

$$ayz + bzx + cxy + (a'x + b'y + c'z)w = 0.$$

The equation of any plane through the intersection of the tangent plane at A with the face BCD is of the form

$$\alpha x + cy + bz + a'w = 0,$$

for any plane containing the line corresponding to B,

$$cx + \beta y + az + b'w = 0,$$

and when these coincide

$$b'/a' = a/b, \ \therefore\ aa' = bb' = cc' = \sigma \text{ suppose.}$$

Let $Ax + By + Cz + Dw = 0$ be the equation of a tangent plane at (x', y', z', w'); proceeding as in XLII. (9)

$$2abcx' = -\sigma aw' + \rho a(-Aa + Bb + Cc), \ \&\text{c.},$$

and from the equations $x'/a + y'/b + z'/c = \rho D/\sigma$ and $Ax' + \dots = 0$ we obtain the quadratic equation

$$(Aa + Bb + Cc - 2abcD/\sigma)^2 = 3(2Bb.Cc + \dots - A^2a^2 - \dots).$$

The three given tangent planes give three equations of the second degree in a, b, c, and therefore eight conicoids.

(4) As in Art. 588, the equation of any sphere is

$$(px+qy+rz+sw)(x+y+z+w)-a'^2yz-\ldots-a''^2xw-\ldots=0,$$

which becomes $px^2+qy^2+rz^2+sw^2=0$ (1), if the fundamental tetrahedron be self-conjugate with respect to the sphere;

$\therefore q+r=a^2$, &c. (2), whence $2p+a^2=b^2+c^2$ and $2p+c'^2=b^2+a'^2$, $\therefore a^2+a'^2=b^2+b'^2=c^2+c'^2$, also $p=bc\cos BAC=a'b\cos CAD$ (3), hence the projections of AB and AD on AC are each equal to AN, AC is therefore perpendicular to the plane BND, and so to BD.

Equations (2) shew that only one of p, q, r, s can be negative, and by (1) one must be so; let p be negative, therefore, by (3), and since also $p=ca'\cos BAD$, each of the angles at A is obtuse.

The sphere meets AB in P and P', where $px^2+qy^2=0$ and $p+q=c^2$, $\therefore q>-p$, $AP/BP=\sqrt{(-p/q)}=AP'/BP'$, $\therefore A$ is within the sphere, which does not intersect BD, DC, or CB; $\therefore B$, C, D are without the sphere.

If R be the radius, O the centre, by Art. 588 $DO^2-R^2=s$, and, by Art. 101, $DO^2=s-(p^{-1}+q^{-1}+r^{-1}+s^{-1})^{-1}$.

(5) Let $ax^2+by^2+cz^2+dw^2=0$ be the equation of the conicoid, P_1, P_2, P_3, P_4 the angular points of a second tetrahedron, and x_1, $y_1\ldots$, $x_2\ldots$, $x_3\ldots$, $x_4\ldots$ their coordinates. The polar of P_2 contains P_1, P_3 and P_4; $\therefore ax_2x_3+\ldots=0$, $ax_2x_4+\ldots=0$, and similarly $ax_3x_4+\ldots=0$. If P_2, P_3, P_4 be given, these determine the ratios $a:b:c:d$. Also x_1, $y_1\ldots$ are given by $ax_2x_1+\ldots=0$, $ax_3x_1+\ldots=0$, and $ax_4x_1+\ldots=0$, $\therefore P_1$ is not in an arbitrary position.

(6) Let the tetrahedron satisfying the conditions be the fundamental tetrahedron for four-point coordinates. The tangential equations of the two conicoids will be

$U\equiv aqr+brp+cpq+a'ps+b'qs+c'rs=0$ and $V\equiv Aqr+A'ps=0$;

using the notation of Art. 568, and the condition $H(\lambda U+\mu V)=0$, see Art. 392, for determining the invariants,

$$\Delta'=A^2A'^2,\quad \Theta'=2AA'(Aa'+A'a)$$
$$\Phi=(Aa'+A'a)^2+2AA'(aa'-bb'-cc'),$$
$$\Theta=2(Aa'+A'a)(aa'-bb'-cc'),$$

whence $\Theta'(4\Delta'\Phi-\Theta'^2)=8\Delta'^2\Theta$ is true for any other fundamental tetrahedron.

(7) The tangential equations of the two conicoids are $p^2/l+\ldots=0$ and $p^2/l'+\ldots=0$, $\therefore \Delta'=(l'm'n'r')^{-1}$,

$$\Theta'=(lm'n'r')^{-1}+(l'mn'r')^{-1}+(l'm'nr')^{-1}+(l'm'n'r)^{-1},$$
$$\Phi=(lmn'r')^{-1}+\ldots,\quad \Theta=(l'mnr)^{-1}+\ldots.$$

If we write l', m', n', r' for l, m, n, r and l'^2/l, m'^2/m, n'^2/n, r'^2/r for l', m', n', r', the new values of Δ', Θ', &c., will be the old ones multiplied by $lmnr/l'm'n'r'$, $\therefore$ the same relations as before hold between these invariants.

(8) Referred to another tetrahedron with parallel faces the equation is $l(x+\alpha)^2+m(y+\beta)^2+n(z+\gamma)^2+r(w+\delta)^2=0$, which when made homogeneous must have no terms in x^2, y^2, z^2, and w^2,

$$\text{hence } l(1+2\alpha)=m(1+2\beta)=n(1+2\gamma)=r(1+2\delta)$$
$$=-l\alpha^2-m\beta^2-n\gamma^2-r\delta^2=k,$$
$$\therefore\ 2\alpha=k/l-1 \text{ \&c.},\quad \therefore\ 4k+l(k/l-1)^2+\ldots=0,$$
$$\therefore\ k^2(l^{-1}+m^{-1}+n^{-1}+r^{-1})-4k+l+m+n+r=0,$$

giving, with the condition, two real values of k.

(9) Taking the tetrahedron as the fundamental one, the equations will be

$$U\equiv ax^2+\ldots+2a'yz+\ldots+2a''xw+\ldots=0,$$
$$V\equiv \alpha x^2+\beta y^2+\gamma z^2+\delta w^2=0.$$

This merely interchanges the U and V of Art. 568, thus Θ' is replaced by Θ, which consequently $=\alpha P+\beta Q+\gamma R+\delta S$ in the notation of that article, and, as shewn there, vanishes.

(10) The equations of the two conicoids must be $lzx+myw=0$ and $nyz+rxw=0$, shew that the discriminant of

$$\mu nyz+\lambda lzx+\mu rxw+\lambda myw=0 \text{ is } \mu^4n^2r^2+\lambda^4l^2m^2-2\lambda^2\mu^2lmnr,$$
$$\therefore\ \Phi^2=4l^2m^2n^2r^2=4\Delta\Delta'.$$

XLIV.

(1) $axdx+\ldots=0$ and $bxdx+\ldots=0$ give the ratios $dx:dy:dz$ or $\xi-x:\eta-y:\zeta-z$. The tangent line at $x=y=z$ is the intersection of the tangent planes $a\xi+b\eta+c\zeta=x^{-1}$ and $b\xi+c\eta+a\zeta=x^{-1}$.

(2) $y^2+2ax=4a^2$, $x^2+z^2=2ax$, take the differentials and write f, g, h for x, y, z, $x-f$ for dx, &c.

For the normal plane $dx:dy:dz=g:-a:-(f-a)g/h$.

(3) $dx:dy:dz=(b-c)a/f:(c-a)b/g:(a-b)c/h$.

(4) Let γ be the constant angle, (r, ϕ) the projection of any point of the curve; $\cot\gamma=dr\operatorname{cosec}\alpha/rd\phi$.

(5) Let $z=ct$, $x=r\cos t$, $y=r\sin t$, $r^2(\cos^2 t/a^2+\sin^2 t/b^2)=1$. Near the point $(a, 0, 0)$ let $x=a+\xi$, then neglecting terms of the order of y^3, $2\xi/a=-y^2/b^2$ and $y/a=z/c$; $\therefore$ the direction-cosines of the osculating plane are as $0:dzd^2\xi:-dyd^2\xi=0:c:-a$; similarly for the point $(0, b, \tfrac{1}{2}\pi c)$ the direction-cosines are as $c:0:-b$.

(6) Take P as the origin and the axes as in Art. 651, $PQ=s$; $QN:PQ-PM=s^3/6\rho\sigma:s^3/6\rho^2=\rho:\sigma$, neglecting higher powers.

For the projection of the tangent at Q on normal plane at P,

$$(\eta-s^2/2\rho)\,s^2/2\rho\sigma-(\zeta-s^3/6\rho\sigma)\,s/\rho=0,$$
$$\text{or } \eta s/2\sigma-\zeta=s^3/4\rho\sigma-s^3/6\rho\sigma=s^3/12\rho\sigma,$$

the shortest distance of the tangents is equal to the perpendicular from the origin on this line $=\frac{1}{2}QN$.

(7) Let ρ be the radius of curvature, x', x'' the first and second differential coefficients of x with respect to t.

$$\rho^{-2}=(x''^2+y''^2+z''^2-s''^2)/s'^4,$$
$$s'^2=4(a^2+b^2t^2), \quad s''=2b^2t(a^2+b^2t^2)^{-\frac{1}{2}}.$$

(8) With the notation of Art. 634, $\rho^{-2}=x''^2+y''^2+z''^2$, $\therefore$ where ρ is a maximum or minimum $x''x'''+\ldots=0$, also $x'x''+\ldots=0$; the direction-cosines of the tangent to the locus of the centre of curvature are, by Art. 637, proportional to $x'+\rho^2x'''$, $y'+\rho^2y'''$, $z'+\rho^2z'''$, and those of the principal normal to x'', y'', z''.

(9) Take the axes as in Art. 651. $2\rho=x^2/y$ ultimately.

i. Turn the axes of y and z through an angle α, so that

$y'=y\cos\alpha+z\sin\alpha$, and $x^2/y'=x^2/y\cos\alpha=2\rho\sec\alpha$ ult.

ii. Turn the axes of z and x through an angle α, so that

$x'=x\cos\alpha+z\sin\alpha$, and $x'^2/y=x^2\cos^2\alpha/y=2\rho\cos^2\alpha$ ult.

XLV.

(1) The planes in which the curves lie are $z=0$, $x=y$, $x+y+z=a$.

The only points of the curve in $z=0$ are the points at infinity in the lines $x\pm y=0$.

The equations of the normal planes at (x, y, z) to the curves in $x=y$ and $x+y+z=a$ are

$$(\xi+\eta-2x)(x-a)(2z+x)-(\zeta-z)(2x+z-a)z=0,$$

and

$$(2z+x)(y+z)\xi-(2z+y)(x+z)\eta+(x-y)z\zeta=(x-y)(3z^2+yz+zx+xy).$$

(2) A principal normal is in the plane of two consecutive elements, if therefore two such normals intersect, three consecutive elements lie in a plane, and therefore the whole curve.

Let $lx+my+nz=p$ be the plane in which the curve lies, $\therefore -la\sin\theta+ma\cos\theta+nf'(\theta)=0$, $\therefore la\sin\theta-ma\cos\theta+nf'''(\theta)=0$; $\therefore f'''(\theta)+f'(\theta)=0$, and $f(\theta)=A+B\sin(\theta+\alpha)$.

(3) If α be the angle at which the helix cuts the generating lines of the cylinder, $x=a\cos\phi$, $y=a\sin\phi$, and $\phi=0$ give the position of the generating point before unwrapping; $ad\phi=ds\sin\alpha$, and $dz=ds\cos\alpha$, $\therefore z=s\cos\alpha$, and the point is in the plane of xy and in the tangent to the circular base, at a distance from the point of contact $=a\phi$, the arc of the circle from which it may be supposed to have been unwrapped.

(4) Using the notation of (3), by Art. 636, $\rho^{-1}=a^{-1}\sin^2\alpha$, and by Art. 637 the coordinates of the centre of curvature are $-a\cot^2\alpha\cos\phi$, $-a\cot^2\alpha\sin\phi$, and $a\phi\cot\alpha$; giving a helix on a cylinder, radius $a\cot^2\alpha$, which will be the same cylinder as for the given helix if $\alpha=\frac{1}{4}\pi$.

(5) With the equations of Art. 597, that of the normal plane is

$$-x\sin\theta+y\cos\theta+nz=n^2a\theta; \quad (1)$$

at the edge of the polar developable, which is the intersection of three consecutive normal planes,

$$-x\cos\theta-y\sin\theta=n^2a, \quad (2) \quad \text{and } x\sin\theta-y\cos\theta=0;$$

$$\therefore z=na\theta, \; x=r\cos\theta, \; y=r\sin\theta, \; r=-n^2a.$$

For the equation of the polar developable, eliminate θ from (1) and (2).

(6) By XLIV. (6) the equation of the projection of the shortest distance is $\eta+\zeta s/2\sigma=0$, hence the angle made with the binormal is $s/2\sigma$, which is half the angle of torsion.

(7) Using the figure for Art. 603, if the element Bbc of the polar surface turn about Bb, until it is in the plane of Aab, r and q will coincide.

$Bq=r$, $Uq=p$, $\sqrt{(r^2-p^2)}=BU$, $\angle VBU=ds/\sigma$, $BU.\angle VBU=dp$.

(8) By Art. 658, the differential equation of the line of greatest slope is $cy\,dx-(x^2+y^2+cx)\,dy=0$, and, if $x=r\cos\theta$, $y=r\sin\theta$, $c\,d\theta+dy=0$, $\therefore y+c\tan^{-1}(y/x)=\text{constant}$.

XLVI.

(1) Let PQ, QR, fig. 2, be two elements of the curve in two plane facets PQL, RQL of the torse, from R draw Rr perpendicular to the plane PQL, Qr is the second element of the curve on the developed torse. Draw RM perpendicular to PQ produced; PQ is perpendicular to Rr and RM, and therefore to rM; the radii of curvature of the curve on the torse and the plane curve are as $\angle rQM : \angle RQM = rM : RM = \cos RMr$, $\angle RMr$ being the angle between the planes PQR and PQL.

(2) For the curve $2z=\sqrt{(x^2+y^2)}=r$, $\therefore$ for the tangent

$$x\xi+y\eta-2r\zeta=r^2-2rz=0, \text{ and } ax\xi+by\eta-2\zeta=2z;$$

for the trace on xy, $x\xi+y\eta=0$, $(a-b)\,x\xi=r$;

$$\therefore (a-b)\,\xi\eta=\pm\sqrt{(\xi^2+\eta^2)}.$$

(3) $x=r\cos\phi$, $y=r\sin\phi$, $r=a\sin\theta$, $z=a\cos\theta$, $a\,d\theta=ds\cos\beta$, $r\,d\phi=ds\sin\beta$, and writing x' for dx/ds, &c.,

$$xy'-yx'=r^2\phi'=r\sin\beta, \; \therefore xy''-yx''=r'\sin\beta;$$

also $x^2+y^2=r^2$, $\therefore\ xx''+yy''=rr''+r'^2-x'^2-y'^2=r(r''-r\phi'^2)$;

$$\therefore\ x''^2+y''^2=(r''-r\phi'^2)^2+r'^2r^{-2}\sin^2\beta,$$

and writing a for r, θ for ϕ, $z''^2+r''^2=(a\theta'^2)^2$;

$$\therefore\ x''^2+y''^2+z''^2=r^{-2}\sin^4\beta-2r''r^{-1}\sin^2\beta+a^{-2}\cos^4\beta+r'^2r^{-2}\sin^2\beta,$$

$$r'=\cos\beta\cos\theta;\ \therefore\ r''=-a^{-1}\cos^2\beta\sin\theta,\ \text{and}\ r''/r=-a^{-1}\cos^2\beta;$$

$$\therefore\ \rho^{-2}=a^{-2}(\sin^4\beta\operatorname{cosec}^2\theta+2\sin^2\beta\cos^2\beta+\cot^2\theta\sin^2\beta\cos^2\beta+\cos^4\beta)$$
$$=a^{-2}(1+\sin^2\beta\cot^2\theta).$$

$\beta=0$ the curve is a meridian, $\rho=a$;

$\beta=\frac{1}{2}\pi$ the curve is a parallel, $\rho=a\sin\theta$.

(4) Let α be the common pitch of the helices; any helix having a pitch $\frac{1}{2}\pi-\alpha$ cutting the generating lines in the opposite direction, will cut all the former helices orthogonally.

(5) Let (λ, μ, ν) be the direction of the normal to the plane containing the perpendicular and the central radius at (x, y, z);

$$\therefore\ \lambda x+\mu y+\nu z=0\ \text{ and }\ \lambda ax+\mu by+\nu cz=0,$$

$$\text{also } \lambda\,dx+\mu\,dy+\nu\,dz=0;$$

shew that $\lambda:\mu:\nu=(b-c)/x:(c-a)/y:(a-b)/z$;

$$\therefore\ (b-c)\,dx/x+(c-a)\,dy/y+(a-b)\,dz/z=0.$$

(6) As in XLIV. (6), the shortest distance is $(\delta s)^3/12\rho\sigma$, and by Art. 643 or 656, $a=\rho\cos^2\alpha=\sigma\sin\alpha\cos\alpha$.

(7) With the axes used in Art. 651 the equation of the plane is $\xi\cos\alpha+\eta\cos\beta+\zeta\cos\gamma=0$, at the projection of P, (x, y, z), $(\xi-x)/\cos\alpha=(\eta-y)/\cos\beta=(\zeta-z)/\cos\gamma=-x\cos\alpha-y\cos\beta-z\cos\gamma$;

$$\therefore\text{, neglecting } s^3,\ \xi=\ \ s\sin^2\alpha-\cos\alpha\cos\beta\, s^2/2\rho,$$
$$\eta=-s\cos\alpha\cos\beta+\sin^2\beta\, s^2/2\rho,$$
$$\zeta=-s\cos\alpha\cos\gamma-\cos\beta\cos\gamma\, s^2/2\rho.$$

The direction-cosines of the tangent at O to the projection are

$$l=\sin\alpha,\ m=-\cot\alpha\cos\beta,\ n=-\cot\alpha\cos\gamma;$$
$$n\eta-m\zeta=-\cos\alpha\cos\gamma\, s^2/2\rho\sin\alpha,$$
$$l\zeta-n\xi=-\cos\beta\cos\gamma\, s^2/2\rho\sin\alpha,$$
$$m\xi-l\eta=-\cos^2\gamma\, s^2/2\rho\sin\alpha;$$

$\therefore$ the perpendicular from P on the tangent $=\cos\gamma\, s^2/2\rho\sin\alpha$,

$$OP^2=\xi^2+\eta^2+\zeta^2=s^2\sin^2\alpha;$$

$\therefore$ the radius of curvature of the projection at $O=\rho\sin^3\alpha/\cos\gamma$.

(8) By Art. 610, the angle of torsion of the curve is equal to the angle of contingence of the locus of the centre of spherical curvature, $\therefore\ ds'/\rho_1=ds/\sigma$, also $ds/\rho=ds'/\sigma_1$.

(9) The locus of the extremities of radii of a sphere drawn parallel to the tangents to the curve is a small circle, hence the tangents are inclined at a constant angle to the radius drawn to the pole of the small circle.

(10) In the figure, p. 251, a circle goes round $BVUq$; $\therefore \angle UBq = \angle UVq$.

XLVII.

(1) Take P, Q, R, S any four consecutive points, O the fixed point; P, Q, R, O and Q, R, S, O lie in the plane QRO, which must therefore contain every point.

Prove that

$$x\,dx/(y^2-z^2) = y\,dy/(z^2-x^2) = z\,dz/(x^2-y^2) = \rho, \quad (1)$$

and that

$$y(x^2-y^2)\,d^2y + z(x^2-z^2)\,d^2z = 3\{(x\,dx)^2+\ldots\} - x^2\{(dx)^2+\ldots\}, \quad (2)$$

$$\text{but } (y^2-z^2)^2 = \tfrac{1}{2}a^4 - x^4 - 2y^2z^2 = x^2(2a^2-3x^2),$$

$$\text{since } 2(y^2z^2+z^2x^2+x^2y^2) = a^4 - \tfrac{1}{2}a^4 = \tfrac{1}{2}a^4;$$

$$\therefore (x\,dx)^2+\ldots = \rho^2(2a^4-\tfrac{3}{2}a^4) = \tfrac{1}{2}a^4\rho^2,$$

$$\text{and } (dx)^2+\ldots = \rho^2(6a^2-3a^2) = 3a^2\rho^2,$$

$$\text{and by (1) and (2) } x^2yz(dz\,d^2y - dy\,d^2z) = \rho^3\{\tfrac{3}{2}a^4x^2 - 3a^2x^4\};$$

$$\therefore xyz\{x(dz\,d^2y - dy\,d^2z)+\ldots\} = 0,$$

hence the osculating plane passes through the origin.

Note. That the intersection consists of plane curves appears by the solution of the equations, for, eliminating z, we obtain $x^2+y^2-\tfrac{1}{2}a^2 = \pm xy$; $\therefore x^2+y^2-z^2 = \pm 2xy$, and $x \mp y \mp z = 0$.

(2) Let $VA, = c$, be the perpendicular on the line AP from the point where the vertex comes, $\angle AVP = \theta$, and let 2α be the angle of the cone; take Vz for the axis of the cone VA in the plane zx, $\theta \operatorname{cosec}\alpha$ is the angle between the projections on xy of VA and VP.

$$VP = c\sec\theta, \quad s = c\tan\theta, \quad x = c\sec\theta\sin\alpha\cos(\theta\operatorname{cosec}\alpha),$$

$$y = c\sec\theta\sin\alpha\sin(\theta\operatorname{cosec}\alpha), \quad z = c\sec\theta\cos\alpha.$$

Prove that
$$d^2x/ds^2 = -(c\sin\alpha)^{-1}\cos^2\alpha\cos^3\theta\cos(\theta\operatorname{cosec}\alpha),$$
$$d^2y/ds^2 = -(c\sin\alpha)^{-1}\cos^2\alpha\cos^3\theta\sin(\theta\operatorname{cosec}\alpha),$$
$$d^2z/ds^2 = c^{-1}\cos\alpha\cos^3\theta,$$

and thence that $\rho = c\tan\alpha\sec^3\theta \propto VP^3$.

(3) For the osculating plane at $(a\cos\theta,\ a\sin\theta,\ \zeta)$

$$(x-a\cos\theta)(\cos\theta\zeta''+\sin\theta\zeta')$$
$$+(y-a\sin\theta)(-\cos\theta\zeta'+\sin\theta\zeta'')+(z-\zeta)a=0,$$

writing ζ', ζ'' for $d\zeta/d\theta$ and $d^2\zeta/d\theta^2$; and for the normal section $x\sin\theta = y\cos\theta$; hence, if λ, μ, $\cos\gamma$ be the direction-cosines of the

line in question, $\lambda/\cos\theta = \mu/\sin\theta = -a\cos\gamma/\zeta''$;

$$\therefore \zeta'' = a\cot\gamma \text{ and } \zeta = \tfrac{1}{2}a\cot\gamma\theta^2 + A\theta + B.$$

For the developed curve if $X = a\theta$, $a\zeta = \frac{1}{2}\cot\gamma X^2 + AX + aB$.

(4) In fig., p. 251, $VU = ds''$, $AB = ds'$ ultimately, and if ρ, R be the radii of curvature,

$$d\rho = VU\sin UVq,\quad dR = AB\sin UBq,\quad d\rho = dR,$$
$$\text{and } \angle UVq = \angle UBq,\quad \therefore ds'' = ds'.$$

(5) Let Oz be the axis of the cylinder, and let the centre of the circle be in Ox, $s = a\phi$, $x = b\cos\theta$, $y = b\sin\theta$, $z = a\cos\phi$, $a\sin\phi = b\theta$; prove that, by Art. 636,

$$\rho^{-2} = b^{-2}\cos^4\phi + a^{-2}\sin^2\phi + a^{-2}\cos^2\phi.$$

(6) Let $l\xi + m\eta + n\zeta = 0$ be the equation of a horizontal plane; the direction-cosines of its intersection with the tangent plane at (x, y, z) are as $ny/b^2 - mz/c^2$, &c.,

$$\therefore dx(ny/b^2 - mz/c^2) + dy(lz/c^2 - nx/a^2) + dz(mx/a^2 - ly/b^2) = 0,$$
$$\text{and } x\,dx/a^2 + y\,dy/b^2 + z\,dz/c^2 = 0;$$

hence, if $u = lx/a^2 + \dots$, $dx/(l/p^2 - ux/a^2) = \dots = p^2 ds/\sqrt{(1 - p^2u^2)}$; $\cos\psi = lpx/a^2 + \dots = pu$, hence, if $d\psi/dx$ be a partial differential coefficient of ψ,

$$-\sin\psi\, d\psi/dx = lp/a^2 - uxp^3/a^4 = \sqrt{(1 - p^2u^2)}\, dx/ds \times p/a^2.$$

(7) The equation of the normal plane is

$$2a(\xi\cos\theta - \eta\sin\theta) + c\zeta = \tfrac{1}{2}(16a^2 + 3c^2)\cos 2\theta + \tfrac{3}{2}c^2.$$

From the two consecutive normal planes, if $16a^2 + 3c^2 = b^2$,

$$2a(\xi\sin\theta + \eta\cos\theta) = b^2\sin 2\theta,$$
$$2a(\xi\cos\theta - \eta\sin\theta) = 2b^2\cos 2\theta,$$
$$\therefore c\zeta = \tfrac{3}{2}c^2 - \tfrac{3}{2}b^2\cos 2\theta = 6(4a^2 + c^2) - 3b^2\cos^2\theta,$$
$$a\xi = b^2\cos^3\theta,\quad a\eta = b^2\sin^3\theta,\quad \therefore \xi^{\frac{2}{3}} + \eta^{\frac{2}{3}} = (b^2/a)^{\frac{2}{3}}.$$

The curve is the intersection of two cylinders, one on a base with four cusps in the plane xy, the other on a semi-cubical parabola in the plane zx.

The two curves are similar if $b^2/4a^2 = b^2/c^2$, or $c = 2a$.

XLVIII.

(1) Let the normal at P to the generating parabola meet the directrix in L, and draw PN perpendicular to the directrix; the radius of curvature of the parabola $= 2SP.PL/PN = 2PL =$ twice the radius of curvature of the perpendicular normal section.

(2) $R^{-1}=\rho^{-1}\cos^2\alpha+\rho'^{-1}\sin^2\alpha$ and $R'^{-1}=\rho^{-1}\sin^2\alpha+\rho'^{-1}\cos^2\alpha$,

$$\therefore\ R^{-1}\cos^2\alpha-R'^{-1}\sin^2\alpha=\rho^{-1}\cos 2\alpha.$$

(3) Where $x=y=z$, each $=a/\sqrt{2}=b$ suppose, let $x=b+\xi$, $y=b+\eta$, $z=b+\zeta$. Shew that ultimately $b(\xi-\eta-2\zeta)=3\zeta^2+2\zeta\eta$, the perpendicular on the tangent plane $=(3\zeta^2+2\zeta\eta)/b\sqrt{6}$ and $\xi^2+\eta^2+\zeta^2=2\eta^2+4\eta\zeta+5\zeta^2$, $\therefore$ the diameter of curvature of the normal section through (ξ,η,ζ)

$$=2\rho=b\sqrt{6}\,(2\eta^2+4\eta\zeta+5\zeta^2)/(3\zeta^2+2\zeta\eta),$$

where ρ is a maximum or minimum the roots of

$$2a\sqrt{3}\,\eta^2-4(\rho-a\sqrt{3})\,\eta\zeta-(6\rho-5a\sqrt{3})\,\zeta^2=0$$

are equal, $\therefore 4(\rho-a\sqrt{3})^2+(6\rho-5a\sqrt{3})\,2a\sqrt{3}=0$,

$$\text{or } 2\rho^2+2\rho a\sqrt{3}-9a^2=0.$$

(4) Let R be the radius of curvature of the normal section through the given tangent, then, by Meunier's theorem,

$$R\cos\psi=\rho',\ <\rho,\ \therefore\ \cos\psi/\rho'=\cos^2\theta/\rho+\sin^2\theta/\rho'.$$

(5) Shew that

$$\rho^{-1}+\rho'^{-1}=P^{-3}\{P^2(u+v+w)-(U^2u+\ldots+2VWu'+\ldots)\}$$

$$\text{and } \frac{d}{dx}\frac{U}{P}=\frac{u}{P}-\frac{U}{P^3}(Uu+Vw'+Wv'),\ \&c.$$

(6) Taking the axes as in Art. 678, the projection of the indicatrix is given by $2z=rx^2+2sxy+ty^2$, where $r=-t$.

(7) Shew that $\log(-p)=(m-1)(\log x-\log z)$,

$$\therefore\ -r/p=(m-1)(x^{-1}-pz^{-1}),\quad s/p=(m-1)\,qz^{-1}.$$

At an umbilic $s/pq=r/(1+p^2)$, $\therefore -z^{-1}(1+p^2)=p(x^{-1}-pz^{-1})$,

$$\therefore\ p=-x/z, \text{ similarly } q=-y/z,\ \therefore\ x=y=z=a/3^{1/m}=b.$$

Near the umbilic, let $x=b+\xi$, $y=b+\eta$, $z=c+\zeta$,

$$\therefore\ mb^{m-1}(\xi+\eta+\zeta)+\tfrac{1}{2}m(m-1)\,b^{m-2}(\xi^2+\eta^2+\zeta^2)+\ldots=0,$$

$$\therefore\ \rho=\text{limit}\,\frac{\sqrt{3}}{2}\,\frac{\xi^2+\eta^2+\zeta^2}{\xi+\eta+\zeta}=\frac{b\sqrt{3}}{m-1}.$$

(8) By Art. 718 Cor., $\rho\rho'=(1+x^2/a^2+y^2/b^2)^2\,ab$.

(9) If (l, m, n) be the direction of a normal, prove that $n^2=l^2+m^2$, or $\cos^2\theta=\sin^2\theta$; the integral curvatures are as the surfaces of a unit sphere cut off by a plane distant $1/\sqrt{2}$ from the centre.

(10) This follows from Art. 296, since, for one of the confocal hyperboloids through the point of contact of any of the planes mentioned in the problem, k is constant.

XLIX.

(1) The general condition is $U^2(v+w)+\ldots-2VWu'-\ldots=0$. Shew that this reduces to

$$(y-z)(V^2+W^2)-aVW-2xU(V-W)=0,$$

and that $V^2+W^2=2x^4+a^2(y+z)^2$, $VW=-x^4$,

$$xU(V-W)=-2ayz\{2x^2-a(y-z)\},$$

$$\therefore (y-z)\{2x^4+a^2(y-z)^2\}+ax^4+8ayzx^2=0;$$

multiply by $y-z$, $\therefore 2y^2z^2+(y-z)^4-yzx^2-8y^2z^2=0$.

(2) Let α, β be the semi-axes of the central section parallel to the tangent plane at any point of the curve, and R the radius of the sphere, then $\alpha\beta p=abc$, $\alpha^2+\beta^2+R^2=a^2+b^2+c^2$, hence, by Art. 720, $\rho\rho'=\alpha^2\beta^2/p^2\propto p^{-4}$, $\rho+\rho'=(\alpha^2+\beta^2)/p\propto p^{-1}$.

(3) With the axes of Art. 678, $2z=x^2/\rho+y^2/\rho'$. Let the generators be inclined at an angle α to Ox, and let R, R', ρ, ρ' be the radii of curvature, then, ultimately, $2Rz=(x\sin\alpha-y\cos\alpha)^2$ is the equation of the enveloping cylinder, and

$$x^2/\rho+y^2/\rho'=(x\sin\alpha-y\cos\alpha)^2/R$$

must give equal values of $x:y$;

$$\therefore R/\rho\rho'=\sin^2\alpha/\rho'+\cos^2\alpha/\rho,$$

and $R'^{-1}=\cos^2\alpha/\rho+\sin^2\alpha/\rho'=R/\rho\rho'$.

(4) At an umbilic, prove as in XLVIII. (7) that $p=-x/z$, $q=-y/z$, and thence that $x^{\frac{2}{3}}/a^{\frac{1}{3}}=y^{\frac{2}{3}}/b^{\frac{1}{3}}=z^{\frac{2}{3}}/c^{\frac{1}{3}}=(a+b+c)^{-1}$; shew also, by comparing the tangent planes to the surface and sphere, that at their point of contact $x^{\frac{2}{3}}/a^{\frac{1}{3}}R=y^{\frac{2}{3}}/b^{\frac{1}{3}}R=z^{\frac{2}{3}}/c^{\frac{1}{3}}R$.

The point of contact is an umbilic if $R^{-1}=a+b+c$.

(5) As in (2) $\rho\rho'=\alpha^2\beta^2/p^2=a^2b^2c^2/p^4$.

(6) The integral curvature is the whole surface of the unit sphere less the two portions included between the two sheets of the cone reciprocal to the asymptotic cone $=4\pi a/\sqrt{(a^2+c^2)}$.

(7) This is to find the envelope of a plane $x\xi/a+y\eta/b+z\zeta/c=0$, subject to the conditions $x^2/a+y^2/b+z^2/c=1$,

and $x^2/(a+k)+y^2/(b+k)+z^2/(c+k)=1$,

its equation is $(a+k)\xi^2/a+(b+k)\eta^2/b+(c+k)\zeta^2/c=0$, which gives a surface meeting the ellipsoid in a sphere.

(8) Let the line of curvature be the intersection of

$$x^2/a+\ldots=1 \text{ and } x^2/(a-k)+\ldots=1,$$

so that $y^2(b-a)/b(b-k)+z^2(c-a)/c(c-k)=1$,

and let α be the inclination of a circular section to the plane of xy; take η, ζ the coordinates in the cyclic plane of the projection, $y=\eta$ and $z=\zeta\sin\alpha$, and $b(a-c)\sin^2\alpha=c(a-b)$; shew that $\zeta^2/(k-c)-\eta^2/(b-k)=b/(a-b)$, a conic the square of half the distance of whose foci $=b(b-c)/(a-b)$; for the umbilics $(a-c)z^2=c(b-c)$ and for their projection $\zeta^2=z^2\operatorname{cosec}^2\alpha=b(b-c)/(a-b)$.

(9) At every point $x=r\cos\theta$, $y=r\sin\theta$, $z=a\theta$, shew that $p=-a\sin\theta/r$, $q=a\cos\theta/r$; then the equations of the normal are

$$r(\xi-r\cos\theta)-a\sin\theta(\zeta-a\theta)=0,\ r(\eta-r\sin\theta)+a\cos\theta(\zeta-a\theta)=0;\ (1)$$

$$\therefore\ \xi\cos\theta+\eta\sin\theta=r,\ r(\xi\sin\theta-\eta\cos\theta)=a(\zeta-a\theta).\ (2)$$

For the lines of curvature consecutive normals intersect,

$$\therefore\ -\xi\sin\theta+\eta\cos\theta=dr/d\theta,\ (3)$$

$$r(\xi\cos\theta+\eta\sin\theta)+(\xi\sin\theta-\eta\cos\theta)\,dr/d\theta=-a^2;$$

$$\therefore\ (dr/d\theta)^2=r^2+a^2,\ \text{and so } r/a=\sinh(\theta+\gamma).$$

By (1) $\rho^2=(a^2/r^2+1)(\zeta-a\theta)^2=(a^2+r^2)^2/a^2$, by (2) and (3), whence, if r be constant, ρ^2 will be constant.

(10) Let the equations of the helix be

$$x=a\cos\theta,\ y=a\sin\theta,\ z=a\theta\tan\alpha,$$

and let the tangent at a point P meet the cylinder, radius b, in the point Q, where $x=b\cos\phi$, $y=b\sin\phi$,

$$\therefore\ (b\cos\phi-a\cos\theta)/-\sin\theta=(b\sin\phi-a\sin\theta)/\cos\theta=z\cot\alpha-a\theta,$$

hence $b\cos(\phi-\theta)=a=b\cos\beta$, $\phi=\theta+\beta$, $z\cot\alpha-a\theta=b\sin\beta$.

For the osculating plane, which is a tangent plane to the surface, $x\sin\theta-y\cos\theta+z\cot\alpha-a\theta=0$ (1); and, for the equation of the surface, eliminate θ from (1) and the equation

$$x\cos\theta+y\sin\theta=a\ (2).$$

Take $(x+\delta x,\ y+\delta y,\ z+\delta z)$, corresponding to $\theta+\delta\theta$, a point in the direction perpendicular to the generating line, so that $-\delta x\sin\theta+\delta y\cos\theta+\delta z\tan\alpha=0$, (3) then 2ρ the principal finite radius of curvature is the limit of

$$\{(\delta x)^2+(\delta y)^2+(\delta z)^2\}/(\delta x\sin\theta-\delta y\cos\theta+\delta z\cot\alpha)\sin\alpha.$$

By (1) and (2), or by the equation of the tangent to the helix at P,

$$x=-(z\cot\alpha-a\theta)\sin\theta+a\cos\theta,$$

$$y=\ \ (z\cot\alpha-a\theta)\cos\theta+a\sin\theta,$$

shew that $\delta x=u\cos\theta+v\sin\theta$, $\delta y=u\sin\theta-v\cos\theta$, where

$$u=-(z\cot\alpha-a\theta)\delta\theta+\tfrac{1}{2}a(\delta\theta)^2,\ v=-\delta z\cot\alpha+\tfrac{1}{2}(z\cot\alpha-a\theta)(\delta\theta)^2\},$$

$$\therefore\ \text{by (3)}\ \delta z\tan\alpha=\tfrac{1}{2}(z\cot\alpha-a\theta)(\delta\theta)^2-\delta z\cot\alpha,$$

$$\therefore\ \text{at } Q\ (\delta x)^2+(\delta y)^2+(\delta z)^2=b^2\sin^2\beta(\delta\theta)^2\ \text{ultimately},$$

$$\text{and } \delta x\sin\theta-\delta y\cos\theta+\delta z\cot\alpha=\tfrac{1}{2}b\sin\beta(\delta\theta)^2,$$

$$\therefore\ \rho=a\tan\beta\operatorname{cosec}\alpha.$$

L.

(1) The squares of the semi-axes of the central section parallel to the tangent plane at (α, β, γ) are given by $a^2\alpha^2/(1-ar^2)+\ldots=0$, Art. 237, and at the centres of curvature which are in the normal,

$$(x-\alpha)/a\alpha=(y-\beta)/b\beta=(z-\gamma)/c\gamma=-r^2, \text{ Art. 721,}$$
$$\therefore\ x=\alpha(1-ar^2),\ y=\beta(1-br^2),\ z=\gamma(1-cr^2),$$
$$\text{and } (x-\alpha)^2\alpha/x+(y-\beta)^2\beta/y+(z-\gamma)^2\gamma/z=0.$$

(2) Transfer the origin to (x, y, z), and writing $\xi+x$, $\eta+y$ for x, y and $p\xi+q\eta+(r\xi^2+2s\xi\eta+t\eta^2)\ldots+z$ for z, the resulting equation must be identically true neglecting powers of ξ, η higher than the second; equating the coefficients of ξ^2, $\xi\eta$, η^2 to zero, each term of the given result is $-b'x-a'y-cz-c''$.

(3) $4(1+p^2+q^2)(rt-s^2)=\{(1+q^2)r-2pqs+(1+p^2)t\}^2$, let $r=\alpha(1+p^2)$, $t=\beta(1+q^2)$, $s=\gamma pq$; prove that
$4(1+p^2+q^2)\alpha\beta=(1+p^2)(1+q^2)(\alpha+\beta)^2-4p^2q^2(\alpha+\beta)\gamma+4p^2q^2\gamma^2=0$,
thence that $(1+p^2+q^2)(\alpha-\beta)^2+p^2q^2(\alpha+\beta-2\gamma)^2=0$, $\therefore\ \alpha=\beta=\gamma$.

(4) Prove that the expression for $\rho\rho'$ in Art. 718, Cor.

$$=\tfrac{1}{4}bc(1+4y^2/b^2+4z^2/c^2)^2=\tfrac{1}{4}bcx^4/p^4.$$

(5) Shew that the angle between the tangent and the axis of $x=\theta$, that $ds/d\theta$ the radius of curvature $=a\cot\theta$, and that the normal cut off by $Ox=y\sec\theta=a\tan\theta$. Hence, the specific curvature is a^{-2}.

(6) The integral curvature is the portion of the surface of the unit sphere included between two parallel planes, whose distances from the centre are $\cos\alpha$ and $\cos\beta$.

(7) Along the curve of contact the normals are perpendicular to the tangent planes of the cone, hence the horograph is formed by the reciprocal cone whose vertical angle is $\pi-2\alpha$.

(8) Let PQR be consecutive points of a line of curvature, Pp, Qq, Rr lines of curvature lying in parallel planes; normals at P and Q intersect in O, at Q and R in O', the plane POQ is perpendicular to the tangent at P to Pp, and therefore to the plane of Pp, $QO'R$ is perpendicular to the plane of Qq, hence POQ and $QO'R$ lie in the same plane, which proves the theorem.

(9) For the paraboloid $pz^{-1}=x^{-1}$, $qz^{-1}=-y^{-1}$, and writing p', q' for p, q in the last two surfaces, viz. $r\pm r'=\text{constant}$ (1),
$xr^{-1}\pm(x+p'z)r'^{-1}$ and $yr^{-1}\pm q'zr'^{-1}=0$, whence $pp'+qq'+1=0$,
and if p'_1, p'_2 be the two values of p', $p'_1p'_2+q'_1q'_2+1=0$.
By (1) the second theorem is true.

(10) Let the given line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n=r$ intersect the conicoid $x^2/a+y^2/b+z^2/c=1$ (1), where $r=r_1$ and r_2; if the normals at these points intersect, they and the given line will lie in one plane $A(x-\alpha)/l+B(y-\beta)/m+C(z-\gamma)/n=0$,

$$\therefore\ A(\alpha+lr_1)/al+\ldots=0 \text{ and } A(\alpha+lr_2)/al=0,$$

$$\text{hence } \alpha A/al+\beta B/bm+\gamma C/cn=0,$$

$$A/a+B/b+C/c=0,$$

$$\text{and } A+B+C=0,$$

$$\therefore\ (b-c)\alpha/l+(c-a)\beta/m+(a-b)\gamma/n=0 \quad (2),$$

and this condition is the same for all the confocals.

Let a chord PR of a confocal to the conicoid (1) touch it in Q. If the normals at P and R intersect, those at the two points which ultimately coincide in Q will also intersect, that is, PQ will be a tangent to a line of curvature on (1). The two conditions to be satisfied are (2) and $(\alpha^2/a+\ldots-1)(l^2/a+\ldots)=(l\alpha/a+\ldots)^2$, giving four directions for PQ.

LI.

(1) The consecutive point to P, $(x, 0, z)$, must be on the normal section perpendicular to the given principal section, the radius of curvature of that normal section is b^2/p, Art. 720, and the centre of curvature $(\xi, 0, \zeta)$ is in the normal at P,

$$\therefore\ (\xi-x)a^2/x=(\zeta-z)c^2/z=-b^2,$$

hence, since $x^2/a^2+z^2/c^2=1$, the locus required is

$$a^2\xi^2/(a^2-b^2)^2+c^2\zeta^2/(b^2-c^2)^2=1. \quad (1)$$

At an umbilic the two centres of principal curvature coincide in a point on the evolute of the principal section. Shew that the equation of the normal at an umbilic is

$$a\xi/\sqrt{(a^2-b^2)}-c\zeta/\sqrt{(b^2-c^2)}=\sqrt{(a^2-c^2)},$$

and hence that the normal touches the ellipse (1) as well as the evolute.

(2) By Art. 710 (3), since $u=v=w=0$,

$$VWu'+WUv'+UWw'=0,$$

$$\text{shew that } u'xyz=ax^2(y+z),\ Ux=ayz, \text{ \&c.},$$

$$\therefore\ VW:WU:UV=x^2:y^2:z^2.$$

(3) Let PG be the normal at P to the generating curve, ψ its inclination to the axis, then $\rho\,.\,PG=a^2$,

$$\therefore\ y\operatorname{cosec}\psi=a^2d\psi/ds \text{ and } y\,dy=a^2\sin\psi\cos\psi\,d\psi,$$

$$\therefore\ y^2=a^2\sin^2\psi+b^2 \text{ and } y\rho=a^2\sin\psi, \text{ where } b \text{ is constant.}$$

If $\psi=0$ when $y=0$, then $b=0$, $\therefore\ \rho=a^2\sin\psi/y=a$.

(4) Let consecutive generators cut Ox in P and P', $OP=x$, $OP'=x+dx$, and let $P'Q=r$ be the distance of a point Q near P' on the generator $P'Q$; the coordinates of Q are

$$\xi = x+\delta x + r\cos(\theta+\delta\theta),\quad \eta = r\sin(\theta+\delta\theta)\cos(\psi+\delta\psi),$$
$$\zeta = r\sin(\theta+\delta\theta)\sin(\psi+\delta\psi).$$

The equation of the tangent plane at P, containing P' and the generator through P, is $\zeta\cos\psi - \eta\sin\psi = 0$, the perpendicular on it from $Q = r\sin(\theta+\delta\theta)\,\delta\psi$, and $PQ^2 = (\delta x + r\cos\theta)^2 + r^2\sin^2\theta$ ult.,

$$\therefore\ 2\rho = \text{limit of } \{(\delta x)^2 + 2r\,\delta x\cos\theta + r^2\}/r\,\delta\psi\sin\theta,$$
$$\therefore\ (dx/d\psi)^2 + 2(\cos\theta\, dx/d\psi - \rho\sin\theta)\,r/\delta\psi + (r/\delta\psi)^2 = 0,$$

which has equal roots, when ρ is a maximum or minimum,

$$\therefore\ (\cos\theta \mp 1)\,dx/d\psi = \rho\sin\theta.$$

(5) Let P, (r,θ,z), be a point in the generator RP, $(r+\delta r,\ \theta+\delta\theta,\ z+\delta z)$ a consecutive point Q; for the tangent plane, which is RPQ ultimately, $\zeta - z = \eta\, dz/r\,d\theta = \eta r^{-1}f'(\theta)$, where η is measured perpendicular to PRz; the perpendicular from Q on the tangent plane for which $\eta = (r+\delta r)\,\delta\theta$, neglecting terms of the third order, and writing u^2 for $r^2 + \{f'(\theta)\}^2$, is

$$\{r\,\delta z - f'(\theta)(r+\delta r)\,\delta\theta\}\,u^{-1} = \{\tfrac{1}{2}rf''(\theta)(\delta\theta)^2 - f'(\theta)\,\delta r\,\delta\theta\}\,u^{-1},$$
$$\therefore\ u^{-1}\rho = \text{limit of } \{(\delta r)^2 + u^2(\delta\theta)^2\}/\{rf''(\theta)(\delta\theta)^2 - 2f'(\theta)\,\delta r\,\delta\theta\},$$
$$\text{hence }\ (dr/d\theta)^2 + 2f'(\theta)\,\rho u^{-1}\,dr/d\theta + u^2 - f''(\theta)\,r\rho u^{-1} = 0$$

gives equal values of $dr/d\theta$, when ρ is a maximum or minimum,

$$\therefore\ \rho_1\rho_2 = -u^4/\{f'(\theta)\}^2.$$

(6) By Art. 718, Cor., $\rho\rho' = (1+p^2+q^2)^2/(rt-s^2)$,

$$pz^{-1} + x^{-1} = 0,\quad rz^{-1} = p^2z^{-2} + x^{-2} = 2x^{-2},\quad sz^{-1} = pqz^{-2} = x^{-2}y^{-2},$$
$$(rt-s^2)\,z^{-2} = 3x^{-2}y^{-2},\quad (1+p^2+q^2)\,z^{-2} = x^{-2}+y^{-2}+z^{-2}.$$

The equation of a tangent plane at (x,y,z) is

$$\xi/x + \eta/y + \zeta/z = 3,$$

therefore, if ϖ be the perpendicular from the origin,

$$9\varpi^{-2} = x^{-2}+y^{-2}+z^{-2},\ \text{ and }\ \rho\rho' = \tfrac{1}{3}x^2y^2z^2.81\varpi^{-4} \propto \varpi^{-4}.$$

At an umbilic, by Art. 719, $x=y=z=(abc)^{\frac{1}{3}}$,

$$\therefore\ 3\varpi^{-2} = x^{-2},\quad \rho\rho' = 3x^2 = 3(abc)^{\frac{2}{3}}.$$

(7) The normals at points in the curve of contact of a circumscribing cylinder are perpendicular to the direction of the axis, hence the horograph for each portion is a great circle.

Hence, the horograph for each of the eight portions of the ellipsoid cut off by the curves of contact is a spherical triangle. Let ABC be one of these triangles, P, Q, R the poles of BC, CA, AB on the hemisphere containing the opposite angles, and let the angles QOR, ROP, POQ be α, β, γ; the angles of the triangle ABC will be $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$; the integral curvature of the

corresponding portion of the ellipsoid will be $2\pi-\alpha-\beta-\gamma$; the remaining portions of the lunes will be $\beta+\gamma-\alpha$, $\gamma+\alpha-\beta$, and $\alpha+\beta-\gamma$; their sum being 2π, which is the area of the hemisphere.

(8) If (x, y, z) be an umbilic, the radii of curvature of all normal sections through it will be equal.

The equation of the tangent plane at (x, y, z) is

$$ax^2(\xi-x)+by^2(\eta-y)+cz^2(\zeta-z)=0,$$

and the perpendicular upon it from an adjacent point, whose coordinates are $x+\lambda s$, $y+\mu s$, $z+\nu s$, is $(ax^2\lambda+\ldots)\,s/\sqrt{(a^2x^4+\ldots)}$,

$$\text{but } a(x+s\lambda)^3+b(y+s\mu)^3+c(z+s\nu)^3=k^3;$$

$$\therefore\ (ax^2\lambda+\ldots)\,s+(ax\lambda^2+\ldots)\,s^2+\ldots=0,$$

hence the radius of curvature of the corresponding normal section

$$=\tfrac{1}{2}\text{ limit } s^2\sqrt{(a^2x^4+\ldots)}/(ax\lambda^2+\ldots)\,s^2,$$

which is independent of the direction (λ, μ, ν) if $ax=by=cz=\sigma$, where $\sigma^3(a^{-2}+b^{-2}+c^{-2})=k^2$, giving the umbilic.

For the normal at the umbilic $(\xi-\sigma a^{-1})/\sigma^2a^{-1}=\ldots=\ldots$;

$$\therefore\ a\xi=b\eta=c\zeta=\sigma',$$

and this must intersect the normal at a consecutive point, whose coordinates are $\sigma a^{-1}+s\lambda$, $\sigma b^{-1}+s\mu$, $\sigma c^{-1}+s\nu$,

$$\therefore\ \frac{(\sigma'-\sigma)\,a^{-1}-s\lambda}{a(\sigma a^{-1}+s\lambda)^2}=\frac{(\sigma'-\sigma)\,b^{-1}-s\mu}{b(\sigma b^{-1}+s\mu)^2}=\frac{(\sigma'-\sigma)\,c^{-1}-s\nu}{c(\sigma c^{-1}+s\nu)^2},$$

$$\therefore\ \frac{b\mu-a\lambda}{2\sigma(a\lambda-b\mu)+s(a^2\lambda^2-b^2\mu^2)}=\frac{c\nu-a\lambda}{2\sigma(a\lambda-c\nu)+s(a^2\lambda^2-c^2\nu^2)},$$

whence $a\lambda=b\mu$, or $a\lambda=c\nu$, or $s(a\lambda+b\mu)=s(a\lambda+c\nu)$, *i.e.* $b\mu=c\nu$, which give the three directions required.

(9) The polar of $(f, 0, 0)$ with respect to one of the confocals $x^2/(a+k)+\ldots=1$, is $fx=a+k$, hence the locus of the points of contact has the equation

$$x/f+y^2/(fx-a+b)+z^2/(fx-a+c)=1; \quad (1)$$

corresponding to the point $(0, g, 0)$, the equation is

$$x^2/(gy-b+a)+y/g+z^2/(gy-b+c)=1; \quad (2)$$

subtracting, we have the two factors

$$\text{i.}\quad gy-b-fx+a=0,$$

$$\text{ii.}\quad (fx-a+b)\,x/f+(gy-b+a)\,y/g+z^2=0.$$

Case i. $x/f+y/g+z^2/(fx-a+c)=1$,

or $x^2+y^2+z^2-(a-c)\,x/f-(b-c)\,y/g=fx-a+c$ or $gy-b+c$, which gives a circular section.

Case ii. $x^2+y^2+z^2-(a-b)(x/f-y/g)=0$,

and, by eliminating z^2 from (1) and (2),

$(c-a)\{x^2-(a-b)/f\}+(c-b)\{y^2+(a-b)/g\}+(fx-a+b)(gy-b+a)=0$,

whence the remainder of the intersection is a sphero-conic.

To prove that the surfaces (1) and (2) cut orthogonally at the circular sections, shew that, writing $fx-a=\sigma=gy-b$, the direction-cosines of the normals at (x, y, z) are as

$$f^{-2}-g^{-2}-z^2/(\sigma+c)^2 : 2/fg : 2z/(\sigma+c)f,$$

$$\text{and } 2/fg : g^{-2}-f^{-2}-z^2/(\sigma+c)^2 : 2z/(\sigma+c)g.$$

Note. The book makes the theorem too general.

(10) At the line of separation the product of the principal radii of curvature changes sign, $\therefore rt-s^2=0$, and one of the radii becomes infinite, Art. 718. Take the origin at any point of the line, Oz the normal, xOz, yOz planes of the principal normal sections, the equation of the surface near the origin is

$$2z=ax^2+bx^3+3cx^2y+3dxy^2+ey^3,$$

since the coefficient of y^2 vanishes; shew that $rt-s^2=3a(dx+ey)+$ terms of higher order, hence $dx+ey=0$ gives the tangent to the boundary; the tangents to the lines of curvature are Ox and Oy, therefore they are not generally tangents to the boundary.

The inflexional tangents are $x^2=0$, and therefore coincide.

LII.

(1) Let the equation of the surface be $z=f(\rho)$, where $\rho^2=x^2+y^2$, the condition gives $(1+q^2)r-2pqs+(1+p^2)t=0$, deduce from this that $f'(\rho)+\rho f''(\rho)+\{f'(\rho)\}^3=0$,

$$\text{and } d\{f'(\rho)\}^{-2}/d\rho=2\rho^{-1}[1+\{f'(\rho)\}^{-2}],$$

$$\text{whence } f'(\rho)=c/\sqrt{(\rho^2-c^2)}, \text{ where } c \text{ is constant};$$

$$\therefore z+\alpha=c\log\{\rho/c+\sqrt{(\rho^2/c^2-1)}\} \text{ and } 2\rho/c=e^{(z+\alpha)/c}+e^{-(z+\alpha)/c}.$$

Geometrically, the catenary is the only curve for which the normal is equal to the radius of curvature in the opposite direction.

(2) Fig. 3. Let PQ, QR be small elements of the plane curve, P, Q being points on consecutive generating lines of the torse, QS the generator through Q; produce PQ to T and draw MRT parallel to QS, QM perpendicular to RT; turn $QMRT$ about PS to the position $QM'R'T'$ in the facet of the torse consecutive to PQS. $\angle RQT$ and $\angle R'QT$ are ultimately the angles of contingence of the plane and bent curves, draw Rt perpendicular to QT and join R', t, therefore $Rt/QR=QR/\rho$, and $R't/QR'=QR'/\rho'$,

$$\text{also } MM'/QM=RR'/QR\sin\theta=QR\sin\theta/R,$$

and since $R't^2=Rt^2+RR'^2$, and $QR'=QR$ ultimately,

$$\rho'^{-2}=\rho^{-2}+\sin^4\theta R^{-2}.$$

(3) Let consecutive generating circles cut Ox in P and Q, $OP=x$, $OQ=x+p\delta\theta$; the tangent plane at P contains Ox and

the tangent to the circle (P), hence its equation is $z\cos\theta+y\sin\theta=0$; take a point R in the circle (Q) near Q, in fig. 4, let Qy', Qz' be parallel to Oy, Oz, and let $\angle QCR$ between the radii CR, CQ be ϕ, if y', z' be coordinates of R,

$$y' = CQ\sin(\theta+\delta\theta) - CR\sin(\theta+\delta\theta+\phi),$$
$$z' = CQ\cos(\theta+\delta\theta) - CR\cos(\theta+\delta\theta+\phi),$$

the perpendicular from R on the tangent plane at P

$$=(r+\delta r)\{\cos\delta\theta-\cos(\delta\theta+\phi)\}=\tfrac{1}{2}r(2\phi\delta\theta+\phi^2) \text{ ultimately,}$$
$$\text{and } PR^2=(p\delta\theta)^2+(r\phi)^2;$$

$\therefore$ ρ the radius of curvature of the normal section through PR

$$=\text{limit of } \{(p\delta\theta)^2+(r\phi)^2\}/r(2\phi\delta\theta+\phi^2);$$

when ρ is a maximum or minimum,

$$(p\delta\theta)^2-2\rho r\phi\delta\theta+(r^2-r\rho)\phi^2=0$$

gives equal values of $\delta\theta/\phi$, $\therefore p^2(r-\rho)=\rho^2 r$.

(4) Let P, Q, fig. 5, be adjacent points on the circle corresponding to ψ and $\psi+\delta\psi$, pP, qQ the corresponding generators, on qQ take a point R near Q, and let $QR=s$; let the axes of x and y be OP and OD perpendicular to OP.

The equation of the tangent plane at P, containing pP and the tangent to the circle at P, is $(x-a)\cos\theta-z\sin\theta=0$.

At R, $x=\{a+s\sin(\theta+\delta\theta)\}\cos\delta\psi$, $z=s\cos(\theta+\delta\theta)$, hence the perpendicular from R on the tangent plane at P is

$$a(1-\cos\delta\psi)\cos\theta-s\sin\delta\theta, \text{ neglecting } s(\delta\psi)^2,$$
$$\text{also } PR^2=(a\delta\psi)^2+s^2;$$

$\therefore$ ρ, the radius of curvature of the normal section through PR,

$$=\text{limit of } \{(a\delta\psi)^2+s^2\}/\{a\cos\theta(\delta\psi)^2-2s\delta\theta\},$$

for the principal curvatures,

$$s^2+2\rho s\delta\psi\, d\theta/d\psi+(a^2-a\rho\cos\theta)(\delta\psi)^2=0$$

gives equal values of $s/\delta\psi$, $\therefore \rho^2(d\theta/d\psi)^2=a^2-a\rho\cos\theta$.

(5) Let Pz be the common normal at P, and let Px, Py bisect the angles between the normal sections of curvature $a+b$ and $a'+b'$; shew that the equations of the two surfaces are

$$2z=(x\cos\tfrac{1}{2}\omega+y\sin\tfrac{1}{2}\omega)^2(a+b)+(x\sin\tfrac{1}{2}\omega-y\cos\tfrac{1}{2}\omega)^2(a-b)+\ldots$$
$$\text{and } 2z=(x\cos\tfrac{1}{2}\omega-y\sin\tfrac{1}{2}\omega)^2(a'+b')+(x\sin\tfrac{1}{2}\omega+y\cos\tfrac{1}{2}\omega)^2(a'-b')+\ldots.$$

At the curve of intersection $Ax^2+2Bxy+Cy^2+\ldots=0$, where $A-C=2(b-b')\cos\omega$, $A+C=2(a-a')$, $2B=2(b+b')\sin\omega$; if $\tan\phi_1$, $\tan\phi_2$ be the values of y/x given by $Ax^2+2Bxy+Cy^2=0$, prove that $(A+C)^2\sec^2(\phi_2\sim\phi_1)=(A-C)^2+4B^2$,

$$\therefore (a-a')^2\sec^2\theta=(b-b')^2\cos^2\omega+(b+b')^2\sin^2\omega.$$

(6) Take the same axes as in (4), and let S be a point in qQ near q, $qS=s$, POp is the tangent plane at p, and the perpendicular on it from $S = s\sin(\theta+\delta\theta)\sin\delta\psi$,

$$pS^2 = pq^2 + 2pq\,.\,s\cos(\theta+\delta\theta) + s^2,$$

where $pq = a\operatorname{cosec}^2\theta\,\delta\theta$ ultimately.

Shew as in (4) that the principal radii of curvature are given by

$$(\rho\sin\theta\,d\psi/d\theta - a\operatorname{cosec}^2\theta\cos\theta)^2 = a^2\operatorname{cosec}^4\theta,$$

and $d\psi/d\theta = \operatorname{cosec}\theta$, $\therefore\ \rho_1\rho_2 = -a^2\operatorname{cosec}^2\theta$, the same as at P.

(7) Take the point P in the plane of zx, and let the two generators cut the principal circular section in Q, Q' subtending an angle 2α at the centre. The direction-cosines of PQ are as $\sin\alpha : -\cos\alpha : 1$, and if (λ, μ, ν) be the direction of the normal at any point of PQ, $\lambda\sin\alpha - \mu\cos\alpha + \nu = 0$, hence the part of the horograph corresponding to PQ is on a great circle, inclined to planes xy and zx at angles $\frac{1}{4}\pi$ and θ, where $\cos\theta = \cos\alpha/\sqrt{2}$, the horograph of the portion included between PQ and the planes zx, xy is a spherical triangle whose spherical excess E is $\frac{1}{4}\pi + \frac{1}{2}\pi + \theta - \pi$, $\therefore\ \sin 2E = -\cos 2\theta = \sin^2\alpha$, but $h = a\tan\alpha$, therefore the integral curvature required, the surface being anticlastic,

$$= -2E = -\sin^{-1}\{h^2/(a^2+h^2)\}.$$

(8) By Art. 718 (3) $(1+q^2)s - pqt = (1+p^2)s - pqr$,

$\therefore\ pq(r-t) + (q^2-p^2)s = 0$, the first integrals of which are

$$x - y\,p/q = f(p/q) \quad (1), \qquad z = \phi(p^2+q^2) \quad (2).$$

For a surface of revolution $z = F(x^2+y^2)$,

$$p = 2xF'(x^2+y^2), \quad q = 2yF'(x^2+y^2),$$

$\therefore\ py - qx = 0$, satisfying (1) when $f(p/q) = 0$,

and $p^2+q^2 = 4(x^2+y^2)\{F'(x^2+y^2)\}^2 = \phi^{-1}(z)$, satisfying (2).

(9) At the point $z' = m\tan^{-1}(y'/x')$, $x' = r'\cos\theta'$, $y' = r'\sin\theta'$, shew that $p = -m\sin\theta'/r'$, $q = m\cos\theta'/r'$, hence that the equations of the normal are

$$x - r'\cos\theta' = (z - m\theta')m\sin\theta'/r', \quad y - r'\sin\theta' = -(z-m\theta')m\cos\theta'/r',$$

or $x\cos\theta' + y\sin\theta' = r'$, $x\sin\theta' - y\cos\theta' = (z - m\theta')m/r'$.

Let $x = r\cos\theta$, $y = r\sin\theta$, then $r\cos(\theta-\theta') = r'$, and

$$r\sin(\theta-\theta') = -m(z-m\theta')/r';$$

therefore, eliminating r', $r^2\sin 2(\theta-\theta') = -2m(z-m\theta')$,

and, if (x, y, z) be the point of intersection of consecutive normals,

$$r^2\cos 2(\theta-\theta') = -m^2 = r^2\cos 2\omega, \quad \therefore\ \theta' = \theta - \omega,$$

$$\therefore\ z = m\theta' + \tfrac{1}{2}m\tan 2(\theta-\theta') = m(\theta-\omega) + \tfrac{1}{2}m\tan 2\omega.$$

(10) Let (l, m, n) be the direction of the generator, (λ, μ, ν) and (λ', μ', ν') those of the tangent planes at A and B, and p, p'

the perpendiculars on them from the centre, so that from the conditions of the problem $l^2+\lambda^2+\lambda'^2=1$, &c. But, by Art. 720, writing a^{-2} for a, &c., the specific curvature at $A=p^2/\alpha^2\beta^2$, and its square root is $p/\alpha\beta \propto p^2 \propto \lambda^2 a^2+\mu^2 b^2-\nu^2 c^2$, and the sum of the square roots of the specific curvature $\propto (\lambda^2+\lambda'^2)\, a^2+\ldots \propto (1-l^2)\, a^2+\ldots$, which is constant for the same generator.

LIII.

(1) When the cone is developed into a plane, the part of the geodesic which surrounds the cone, and is terminated by the multiple point, forms the base of an isosceles triangle, and the angle at the vertex is 2β, where $2\beta l=2\pi l \sin\alpha$; but $\cos 2\alpha=\frac{7}{8}$, $\therefore$ $\sin\alpha=\frac{1}{4}$, hence $2\beta=\frac{1}{2}\pi$, and the two branches are each inclined at $\frac{1}{4}\pi$ to the generator through the multiple point.

(2) The principal normal of the curve coincides with the normal to both surfaces at every point.

(3) By Art. 762, pD, which is ac at the umbilic, is the same at the extremity of the mean axis, therefore D at that point $=ac/b$, and is inclined to the plane of a, b at an angle θ, for which

$$D^{-2}=a^{-2}\cos^2\theta+c^{-2}\sin^2\theta, \text{ or } b^2=a^2\sin^2\theta+c^2\cos^2\theta,$$

$$\therefore\ b^2+k=(a^2+k)\sin^2\theta+(c^2+k)\cos^2\theta,$$

or θ is the same for all the confocals, hence the geodesics all touch one of the planes $z=\pm x\tan\theta$.

(4) Follows from Art. 770, since the mean axis bisects the geodesic passing through it and opposite umbilics.

(5) If $d\chi$ be the angle between consecutive principal normals to the geodesic, by Art. 647, $(d\chi/ds)^2=\rho^{-2}+\sigma^{-2}$, but $d\chi$ is also the angle between consecutive normals to the surface, whose equation may be written $2z=x^2/\rho_1+y^2/\rho_2$ ultimately, whence

$$p=x/\rho_1,\quad q=y/\rho_2, \text{ and } \sec^2(d\chi)=1+x^2/\rho_1^{\,2}+y^2/\rho_2^{\,2},$$

$$\therefore\ (d\chi)^2=x^2/\rho_1^{\,2}+y^2/\rho_2^{\,2}=(ds)^2(\cos^2\theta/\rho_1^{\,2}+\sin^2\theta/\rho_2^{\,2}),$$

$$\text{and } \rho^{-2}+\sigma^{-2}=\cos^2\theta/\rho_1^{\,2}+\sin^2\theta/\rho_2^{\,2}, \text{ also } \rho^{-1}=\cos^2\theta/\rho_1+\sin^2\theta/\rho_2,$$

from which eliminate θ.

(6) Fig. 6. Let Aa, Bb, Cc be consecutive generators of the torse, PQ, QR elements of the geodesic in the facets aAb, bBc, so that $\angle PQb=\angle BQR$; $\angle PQb=\psi$, $\angle QRc=\psi+d\psi$, $AB/d\psi=\rho$, $AQ=t$, $BR=t+dt$, $\therefore (t+\rho d\psi)\sin\psi=(t+dt)\sin(\psi+d\psi)$,

$$\therefore\ \rho\, d\psi \sin\psi=t\cos\psi\, d\psi+dt\sin\psi.$$

(7) The geodesics make equal angles with the lines of curvature, and if $\frac{1}{2}\theta$ be the inclination of an umbilical geodesic to the line of curvature corresponding to k_1 of Art. 764,

$$k_2^2 \cos^2\tfrac{1}{2}\theta + k_1^2 \sin^2\tfrac{1}{2}\theta = b^2, \text{ or } (k_2^2 - k_1^2)\cos\theta = 2b^2 - k_2^2 - k_1^2,$$

where k_1^2, k_2^2 are the roots of

$$x^2/a^2(a^2-k^2) + y^2/b^2(b^2-k^2) + z^2/c^2(c^2-k^2) = 0.$$

Let $b^2 - k_1^2 = h_1$, $b^2 - k_2^2 = h_2$, then $(h_1 - h_2)\cos\theta = h_1 + h_2$, and $(h_1+h_2)^2\tan^2\theta = -4h_1h_2$, where h_1, h_2 are roots of

$$(c^2-b^2+h)h\,x^2/a^2 + (a^2-b^2+h)h\,z^2/c^2 + (c^2-b^2+h)(a^2-b^2+h)\,y^2/b^2 = 0,$$

$$\therefore h_1 + h_2 = (b^2-c^2)x^2/a^2 - (a^2-b^2)z^2/c^2 - (a^2+c^2-2b^2)y^2/b^2$$
$$= x^2+y^2+z^2 - (a^2+c^2-b^2)(x^2/a^2 + y^2/b^2 + z^2/c^2),$$

and $\quad h_1h_2 = -(a^2-b^2)(b^2-c^2)y^2/b^2.$

(8) With the notation of Art. 751, $\rho_1 = \infty$, $\theta =$ inclination of the geodesic to the generating line crossed, $\rho^{-1} = \rho_2^{-1}\sin^2\theta$, and $\sigma^{-1} = \rho_2^{-1}\cos\theta\sin\theta$, $\therefore \sigma/\rho = \tan\theta$.

LIV.

(1) Let $x = 0$ be the plane of the geodesic curve; since $x''/U = y''/V = z''/W$, Art. 740, and $x'' = 0$, either, i. $y'' = 0$ and $z'' = 0$, $\therefore y'/z'$ is constant and the geodesic rectilinear, and therefore a generator; or, ii. $U = 0$, that is, the surface is cylindrical.

(2) For an umbilical geodesic $pD = ac$, and at the mean axis $p = b$, $\therefore$, in Art. 751, $\rho_1 = a^2/b$, $\rho_2 = c^2/b$, and $b^2 = c^2\cos^2\theta + a^2\sin^2\theta$.

(3) The condition gives $a^2 + c^2 = 2b^2$; for any umbilical geodesic, Art. 764, $k_1^2\cos^2\theta + k_2^2\sin^2\theta = b^2$, and for any point in one of the circular sections

$$r = b, \therefore k_1^2 + k_2^2 = a^2 + b^2 + c^2 - r^2 = 2b^2 \text{ and } (k_1^2 - k_2^2)\cos 2\theta = 0.$$

(4) With the notation of Arts. 784, 785, $\tan\frac{1}{2}\omega' = m^{-2}\tan\frac{1}{2}\beta = m^{-1}$ and, if θ_2 be the inclination required, $\cot\frac{1}{2}\theta_2 = m\cot\frac{1}{2}\omega' = m^2$.

(5) Let O be the centre of the unit sphere, and suppose B indefinitely near to A, the generating line through A of the torse along AB is parallel to the intersection of planes perpendicular to Oa and Ob, and therefore perpendicular to the tangent to ab at a, similarly for the torse along AC.

(6) Let $\angle QPQ' = 2\alpha$, $\angle UPV = 2\beta$, and, with the notation of Art. 764, let k_2 determine the line of curvature on which P lies, $\therefore k_1^2\cos^2\alpha + k_2^2\sin^2\alpha = \lambda^2$ a constant, Art. 766, for all positions of P,

$$\text{also } k_1^2\cos^2\beta + k_2^2\sin^2\beta = b^2,$$

$$\therefore \cos^2\alpha : \cos^2\beta = \lambda^2 - k_2^2 : b^2 - k_2^2, \ k_2 \text{ being constant.}$$

(7) With the notation of Arts. 751, 764

$$\sigma^{-1}=\sin\theta\cos\theta\,(p/k_1^2-p/k_2^2) \text{ and } k_1^2\cos^2\theta+k_2^2\sin^2\theta=q^2,$$

$$\therefore\ \sin^2\theta/(q^2-k_1^2)=\cos^2\theta/(k_2^2-q^2),$$

$$\therefore\ (k_2^2-k_1^2)\sin\theta\cos\theta=\sqrt{\{(k_1^2+k_2^2)\,q^2-q^4-k_1^2k_2^2\}}.$$

Let OQ be the semi-diameter D and $OR=D'$ conjugate to POQ, q, q' perpendiculars on the tangents at R and Q to the section QOR, $\therefore k_1^2+k_2^2=D^2+D'^2$ and $k_1k_2=Dq=D'q'$, and $(k_1^2+k_2^2)\,q^2-q^4-k_1^2k_2^2=q^2(D^2+D'^2)-q^4-q^2D^2=q^2(D'^2-q^2)$,

$$\therefore\ (k_2^2-k_1^2)\sin\theta\cos\theta=qD'\cos(q,q').$$

Let p' be the perpendicular from O on the tangent plane at Q, q is perpendicular to the osculating plane of the geodesic, which is parallel to the plane containing OQ and p', $\therefore A'=A''\cos(p',q)$, where $A''p'=Ap$, also $p'=q'\cos(p',q')$ and, by spherical trigonometry, $\cos(p',q)=\cos(p',q')\cos(q,q')$;

$$\therefore\ \sigma^{-1}=pqD'\cos(q,q')/k_1^2k_2^2=p\cos(q,q')/q'D$$
$$=p\cos(p',q)/p'D=A''\cos(p',q)/AD=A'/AD.$$

(8) Shew that $\dfrac{x''}{-a\sin\theta/r}=\dfrac{y''}{a\cos\theta/r}=\dfrac{z''}{-1}=\dfrac{1}{a}\dfrac{d}{ds}(r^2\theta')=-a\theta''$,

$\therefore (r^2+a^2)\,\theta'$ is constant, $\cos\psi=r'$, $\sin\psi=\sqrt{(r^2+a^2)}\,\theta'$; the tangent plane contains the generator and the tangent to the geodesic, $\therefore \tan\phi=r\theta'/z'=r/a$, $\sec\phi=\sqrt{(r^2+a^2)}/a$.

(9) Let $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=R^2$ be the equation of the sphere, $\therefore (x-\alpha)\,x''+(y-\beta)\,y''+(z-\gamma)\,z''=-x'^2-y'^2-z'^2=-1$, hence, Art. 740,

$$(x-\alpha)\,U+(y-\beta)\,V+(z-\gamma)\,W=\rho\,(U^2+V^2+W^2)^{\frac{1}{2}},$$

$\therefore \rho$ is the perpendicular from (α,β,γ) on the tangent plane.

Geometrically. The osculating plane at P is the plane of a small circle of the sphere, whose radius is the radius of curvature of the geodesic, and whose centre is at the same distance from the tangent plane as the centre of the sphere.

(10) Let r, r', r'' be the distances from the axis of the angles A, B, C of the geodesic triangle, OA, OB, OC meridians through these angles, then, by Art. 761,

$$r'\sin OBC=r''\sin OCB,\quad r''\sin OCA=r\sin OAC,$$
$$\text{and}\quad r\sin OAB=r'\sin OBA,$$
$$\therefore\ \sin OBC\sin OCA\sin OAB=\sin OCB\sin OAC\sin OBA.$$

LV.

(1) By Art. 778, for an umbilical geodesic $k'\sin^2\tfrac{1}{2}\theta+k''\cos^2\tfrac{1}{2}\theta=b$, where k' and k'' are given by the equation

$$(c-k)\,y^2/b+(b-k)\,z^2/c+(c-k)(b-k)=0,$$

$\therefore\ \tan^2\frac{1}{2}\theta = -(k''-b)/(k'-b) = -h''/h'$ and $\tan^2\theta = -4h'h''/(h'+h'')^2$, h' and h'' being the roots of $(c-b-h)y^2/b - hz^2/c + h(h+b-c) = 0$,

$$\therefore\ h'+h'' = 2x - b + c,\quad -h'h'' = (b-c)\,y^2/b.$$

(2) Writing λ^2 for q^2 in Art. 764, $k_1^2\cos^2\theta + k_2^2\sin^2\theta = \lambda^2$,

$$\therefore\ (k_2^2 - k_1^2)^2\sin^2\theta\cos^2\theta = (k_1^2 + k_2^2)\,\lambda^2 - \lambda^4 - k_1^2k_2^2$$
$$= (a^2+b^2+c^2-x^2-y^2-z^2-\lambda^2)\,\lambda^2 - a^2b^2c^2/p^2,$$

and $p^2k_1^2k_2^2\sigma^{-1} = p^3(k_2^2 \sim k_1^2)\sin\theta\cos\theta$, Art. 751.

(3) With the notation of Art. 774, for two perpendicular geodesic tangents to the same line of curvature, $k_1^2 + k_2^2 = 2q^2$, where $x^2/(a^2-q^2) + y^2/(b^2-q^2) + z^2/(c^2-q^2) = 1$ is the equation of the hyperboloid which determines the line of curvature; if r be the distance of the point P from which the tangents are drawn,

$$a^2+b^2+c^2 = k_1^2 + k_2^2 + r^2 = 2q^2 + r^2;$$

and if one of the principal sections of the hyperboloid be a rectangular hyperbola, $2q^2 = b^2 + c^2$, $c^2 + a^2$, or $a^2 + b^2$, $\therefore r^2 = a^2$, b^2, or c^2, hence P must be one of the extremities of the greatest or least axis, or it must lie on one of the central circular sections.

(4) Shew that $yx'' - xy'' = 0$ or $r^2\theta' = c$,

$$\therefore\ 1 = x'^2 + y'^2 + z'^2 = r'^2 + r^2\theta'^2 + 4a^6r'^2/r^6$$
$$= c^2p^{-2} + 4a^6c^2(p^{-2} - r^{-2})/r^6,$$
$$\therefore\ c^2r^2p^{-2}(r^6 + 4a^6) = r^8 + 4a^6c^2.$$

(5) Let the equation of the right conoid be $z = f(y/x)$, prove that for the geodesic $xx'' + yy'' = 0$; and since $q = xx' + yy'$, $dq/ds = x'^2 + y'^2 = (d\sigma/ds)^2$, $\therefore dq/d\sigma = d\sigma/ds$.

(6) As in Art. 764, $\cos^2\theta\,k_2^{-2} + \sin^2\theta\,k_1^{-2} = D^{-2}$;

$$\text{if } \theta = \tfrac{1}{4}\pi,\quad p\,(k_2^{-2} + k_1^{-2}) \propto pD^{-2} \propto p^3.$$

(7) As in LIV (8) $\sin\psi = \sqrt{(r^2+a^2)}\,\theta' = c/\sqrt{(r^2+a^2)}$, let r_1, r_2, r_3 be the values of r at A, B and C, and c_1, c_2, c_3 the values of c for the geodesics BC, CA, AB, then $\sin\alpha_1 = c_2/\sqrt{\{(r_1^2+a^2)}$, $\sin\alpha_2 = c_3/\sqrt{(r_1^2+a^2)}$, and $\sin\alpha_1/c_2 = \sin\alpha_2/c_3$. Similarly

$$\sin\beta_1/c_3 = \sin\beta_2/c_1 \text{ and } \sin\gamma_1/c_1 = \sin\gamma_2/c_2.$$

(8) The direction of a geodesic at the point $P(x, y, z)$ of an ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2$ is (x', y', z'), where $x' = dx/ds$, &c. The equation of the perpendicular tangent plane is

$$x'x + y'y + z'z = \sqrt{(a^2x'^2 + b^2y'^2 + c^2z'^2)};$$

let the tangent PQ meet this plane in Q and draw OY perpendicular to PQ from the centre, then $OQ^2 = YQ^2 + OY^2 = YQ^2 + OP^2 - PY^2$

$$= a^2x'^2 + b^2y'^2 + c^2z'^2 + x^2 + y^2 + z^2 - (x'x + y'y + z'z)^2.$$

By the geodesic equations, noticing that $x'^2+y'^2+z'^2=1$,

$$\frac{x''}{x/a^2}=\frac{y''}{y/b^2}=\frac{z''}{z/c^2}=\frac{a^2x'x''+\dots}{xx'+\dots}=\frac{xx''+\dots}{1},$$

$$\therefore\ a^2x'x''+\dots=(xx'+\dots)(xx''+\dots),$$

$$\therefore\ a^2x'x''+\dots+xx'+\dots-(xx'+\dots)\,d(xx'+\dots)/ds=0,$$

$$\therefore\ \tfrac{1}{2}d\,OQ^2/ds=0,\ \text{or}\ OQ\ \text{is constant.}$$

This can also be deduced from Joachimsthal's theorem, for since $xx'/a^2+\dots=0$, $-2y'z'yz=b^2c^2(y'^2y^2/b^4+z'^2z^2/c^4-x'^2x^2/a^4)$,

$$\therefore\ (y'z-z'y)^2=y'^2(z^2+c^2y^2/b^2)+z'^2(y^2+b^2z^2/c^2)-b^2c^2x'^2x^2/a^4$$
$$=b^2c^2\{(1-x^2/a^2)(y'^2/b^2+z'^2/c^2)-x'^2x^2/a^4\}$$
$$=c^2y'^2+b^2z'^2-a^2b^2c^2D^{-2}x^2/a^4,$$

$$\therefore\ OQ^2=a^2x'^2+b^2y'^2+c^2z'^2+(y'z-z'y)^2+(z'x-x'z)^2+(x'y-y'x)^2$$
$$=a^2+b^2+c^2-a^2b^2c^2/p^2D^2.$$

LVI.

(1) Take the origin in the vertex of the cone, and let AP be an arc s of the geodesic, PT, a tangent at $P(x, y, z)$, $=s+c$, ξ, η, ζ coordinates of T, $\therefore\ \xi=x-(s+c)x'$, $\xi'=-(s+c)x''$, &c.

If $F=0$ be the equation of the conical surface, $x\,dF/dx+\dots=0$,

$$\therefore\ xx''+\dots=0,\ \text{also}\ x'x''+\dots=0,$$

$$\therefore\ \xi\xi'+\dots=0\ \text{and}\ \xi^2+\eta^2+\zeta^2=\text{constant, proving i.}$$

$$\text{Also}\ \xi x''+\dots=0\ \text{and}\ x'\xi'+\dots=0,$$

$\therefore\ x'\xi+\dots=\text{constant}$, proving ii. and $\xi x''+\dots=0$, proves iii.

(2) The directions of the normal at $P(x, y, z)$, the tangent to the geodesic, and the perpendicular to both, are $(px/a^2, py/b^2, pz/c^2)$, (x', y', z'), and (l, m, n), hence, by Art. 146,

$$x'=mpz/c^2\sim npy/b^2\ \text{and}\ \lambda^{-4}=(x'^2/a^2+\dots)p^{-2},$$

$$\therefore\ a^2b^2c^2/\lambda^4=b^2c^2(mz/c^2-ny/b^2)^2+c^2a^2(nx/a^2-lz/c^2)^2+a^2b^2(ly/b^2-mx/a^2)^2$$
$$=l^2a^2+m^2b^2+n^2c^2-l^2x^2-m^2y^2-n^2z^2-2mnyz-\dots,$$

$$\therefore\ (l^2+m^2+n^2)\,a^2b^2c^2/\lambda^4=l^2a^2+m^2b^2+n^2c^2-(lx+my+nz)^2. \quad (1)$$

But, (l, m, n) being the direction of the generating line of the scroll, $lx+my+nz=p'$, and $l^2a^2+m^2b^2+n^2c^2=p''^2$; $\therefore$ (1) gives the theorem as corrected in the errata.

(3) Let PT, QT be geodesic tangents to an arc PQ of the curve, δu the angle between the tangents which is ultimately the angle of geodesic contingence; by Art. 761, $r\sin\theta$ is constant throughout the geodesic; if r', θ' be the values of r and θ at T in PT, r', $\theta'+\delta u$ are those at T in QT,

$$\therefore\ r'\sin\theta'=r\sin\theta\ \text{and}\ r'\sin(\theta'+\delta u)=r\sin\theta+\delta(r\sin\theta),$$

$$\text{or}\ \sin(\theta'+\delta u)/\sin\theta'=1+\delta(r\sin\theta)/r\sin\theta;$$

$$\therefore\ \text{ultimately}\ du\cot\theta=d(r\sin\theta)/r\sin\theta.$$

(4) The cusp is where $\psi=0$, and the tangent at the cusp is the axis of the generating curve; take Oz the axis of revolution, at a distance $2c$ from the cusp, and let (r, θ, z) be any point in the surface; $dr = ds \cos\psi = 2c \tan\psi \sec\psi\, d\psi$;

$$\therefore\ r = 2c \sec\psi, \text{ also } dz = dr \tan\psi,$$

$$\text{hence } 1 = r^2\theta'^2 + r'^2 + z'^2 = r^2\theta'^2 + r'^2 \sec^2\psi,$$

and by the property of a geodesic on a surface of revolution

$$r^2\theta' = a,\ \therefore\ r^4\theta'^2/a^2 = r^2\theta'^2 + r'^2r^2/4c^2,$$

$$\therefore\ (dr/d\theta)^2 = 4c^2(r^2/a^2 - 1),\ \therefore\ e^{2c\theta/a} + e^{-2c\theta/a} = 2r/a.$$

(5) At any point (ρ, ϕ, z) of the surface, $2\rho/c = e^{z/c} + e^{-z/c}$, whence $dz = c\,d\rho/\sqrt{(\rho^2 - c^2)}$.

For any curve traced on the surface, if $\phi' = d\phi/ds$, &c.,

$$1 = \rho^2\phi'^2 + \rho'^2 + z'^2 = \rho^2\phi'^2 + \rho^2\rho'^2/(\rho^2 - c^2),$$

and for a geodesic $\rho^2\phi' = \text{constant} = kc$ suppose,

$$\therefore\ \rho^2\phi'^2 = k^2c^2\{\phi'^2 + \rho'^2/(\rho^2 - c^2)\},$$

$$\text{hence, for the projection, } (d\phi/d\rho)^2(\rho^2 - c^2)(\rho^2 - k^2c^2) = k^2c^2,$$

$$\text{let } c = \rho \sin\lambda,\quad d\phi = k\,d\lambda/\sqrt{(1 - k^2\sin^2\lambda)},$$

$$\therefore\ \lambda = \operatorname{am}(\phi/k, k),\quad c = \rho \operatorname{sn}(\phi/k, k).$$

(6) Let the equation of the spheroid be $r^2/a^2 + z^2/c^2 = 1$, $\therefore\ rr'/a^2 + zz'/c^2 = 0$, where r' denotes dr/ds, &c., and for a geodesic, $r^2\theta' = b$, a constant;

$$1 = r^2\theta'^2 + r'^2 + z'^2 \text{ and } r'^2/\theta'^2 = (dr/d\theta)^2 = r^4/p^2 - r^2,$$

$$\therefore\ r^4/b^2 = r^2 + (r^4/p^2 - r^2)(1 + c^4r^2/a^4z^2),$$

$$\text{or } p^2(a^2 - r^2 + b^2c^2/a^2) = b^2\{a^2 - r^2(1 - c^2/a^2)\}\ \ (1).$$

Let C be the centre of the elliptic base of a cone whose vertex is V and axis VC, and let CY be perpendicular on the tangent PY to the ellipse, VY is perpendicular to PY; PY will be the tangent to the curve traced by the perimeter of the ellipse on the plane on which the cone rolls, and if $VP=r$, $VY=p$, $VC=h$, and α, β be the semi-axes of the ellipse,

$$CP^2 = r^2 - h^2 \text{ and } CY^2 = p^2 - h^2,$$

$$\therefore\ (p^2 - h^2)(\alpha^2 + \beta^2 + h^2 - r^2) = \alpha^2\beta^2,$$

$$\text{or } p^2(\alpha^2 + \beta^2 + h^2 - r^2) = \alpha^2\beta^2 + h^2(\alpha^2 + \beta^2 + h^2) - h^2r^2,$$

which is of the same form as (1), and α, β, h can be found so that the curves are the same.

(7) Let (r, θ, z) be the point S in cylindrical coordinates, where S is the focus of the ellipse rolling on OZ and touching it at P, ϕ the inclination of SP to OZ; the tangent to the roulette is perpendicular to SP, $\therefore\ dz = dr \tan\phi$; and, writing r' for dr/ds, &c.,

$$1 = r^2\theta'^2 + r'^2 + z'^2 = r^2\theta'^2 + r'^2\sec^2\phi. \quad (1)$$

By the geodesic property $r^2\theta' = \text{constant} = \alpha \sin\gamma$, (2); also $\frac{1}{2}(\alpha+\beta)$ and $\sqrt{(\alpha\beta)}$ being the semi-axes of the ellipse,

$$\alpha\beta/r^2 = (\alpha+\beta)/SP - 1;\ \therefore\ (\alpha+\beta)\,r^{-1}\sin\phi = \alpha\beta r^{-2}+1,$$

$$(\alpha+\beta)^2 r^{-2}\cos^2\phi = (\alpha+\beta)^2 r^{-2} - (\alpha\beta r^{-2}+1)^2 = \alpha^2\beta^2(r^{-2}-\alpha^{-2})(\beta^{-2}-r^{-2}),$$

by (1) and (2),

$$(d\theta)^2\{r^4(\alpha\sin\gamma)^{-2} - r^2\} = (dr)^2(\alpha^{-1}+\beta^{-1})^2 r^{-2}/(r^{-2}-\alpha^{-2})(\beta^{-2}-r^{-2}).$$

Let $\sin^2\psi\,(\beta^{-2}-\alpha^{-2}) = r^{-2}-\alpha^{-2}$, $\therefore \cos^2\psi\,(\beta^{-2}-\alpha^{-2}) = \beta^{-2}-r^{-2}$,

$$\sin\psi\cos\psi\,d\psi\,(\beta^{-2}-\alpha^{-2}) = -dr\,.\,r^{-3};$$

$$\therefore\ (d\theta)^2\{\alpha^{-2}\cot^2\gamma - (\beta^{-2}-\alpha^{-2})\sin^2\psi\} = (\alpha^{-1}+\beta^{-1})^2(d\psi)^2,$$

and $\mu\theta = \int d\psi/\sqrt{(1-k^2\sin^2\psi)}$; $\therefore r^{-2} = \alpha^{-2}\,\mathrm{cn}^2(\mu\theta) + \beta^{-2}\,\mathrm{sn}^2(\mu\theta)$.

(8) For the helicoid let $x = r\cos\theta$, $y = r\sin\theta$, $z = m\theta$, hence, for the geodesic, $x''/my = y''/-mx = z''/r^2$, $\therefore xy''-yx''+mz''=0$, and $(r^2+m^2)\,\theta' = b$, see problem LIV. (8);

$$\therefore\ (r^2+m^2)^2\theta'^2/b = 1 = x'^2+y'^2+z'^2 = r'^2+(r^2+m^2)\,\theta'^2,$$

$$\text{and } (dr/d\theta)^2 = (r'/\theta')^2 = (r^2+m^2)(r^2+m^2-b^2)/b^2.$$

i. $b<m$, let $r = m\cot\psi$, $dr/\sqrt{(r^2+m^2)} = -\operatorname{cosec}\psi\,d\psi$;

$$\therefore\ (\alpha-\theta)/k = \int d\psi/\sqrt{(1-k^2\sin^2\psi)},\ k = b/m,$$

hence $r\,\mathrm{tn}\{(\alpha-\theta)/k\} = m$.

ii. $b>m$, let $r = \sqrt{(b^2-m^2)}\sec\psi$, $dr/\sqrt{(r^2+m^2-b^2)} = \sec\psi\,d\psi$, $\therefore\ \theta-\alpha = \int b\sec\psi\,d\psi/\sqrt{(b^2\sec^2\psi - m^2\tan^2\psi)} = \int d\psi/\sqrt{(1-k^2\sin^2\psi)}$, where $k = m/b$, $\therefore r\,\mathrm{cn}(\theta-\alpha) = m\sqrt{(1-k^2)}/k$.

LVII.

(1) Writing β and γ for a^2-b^2 and a^2-c^2, and eliminating θ,

$$\frac{2x^2/a^2}{\beta+\gamma} = \frac{2y^2/b^2-1}{-\gamma} = \frac{2z^2/c^2-1}{-\beta} = \frac{2(y^2/b^2-z^2/c^2)}{\beta-\gamma};$$

$$\therefore\ x^2/a^2+y^2/b^2+z^2/c^2 = 1. \quad (1)$$

Let $k^2 = \frac{1}{2}(b^2+c^2)$, $\frac{1}{2}(\beta+\gamma) = a^2-k^2$, and $\frac{1}{2}(\gamma-\beta) = b^2-k^2 = -(c^2-k^2)$;

$$\therefore\ x^2/a^2(a^2-k^2)+y^2/b^2(b^2-k^2)+z^2/c^2(c^2-k^2) = 0. \quad (2)$$

Multiply (2) by k^2 and add to (1), $\therefore x^2/(a^2-k^2)+\ldots = 1$.

(2) Using the notation of Art. 810, shew that

$$\sin\alpha = l(\lambda'+\lambda'_1 dp)+\ldots = (l\lambda'_1+\ldots)\,dp;$$

$$\therefore\ \alpha = -TPdp, \text{ and similarly } \beta = -TQdq \text{ ultimately.}$$

(3) By Art. 803 the condition that the curves (p) may be chief curves is $A\alpha''+B\beta''+C\gamma''=0$, and that the same curves should be geodesic, the condition is, by Art. 805, $(Fa'-Ga)\,\alpha''+\ldots=0$, the two conditions not being generally the same.

(4) With the curvilinear coordinates used in Art. 831,

$$(ds)^2 = \beta^2 (dp)^2 + (dq)^2,$$

β being a function of p only, therefore in the equation of a geodesic line in Art. 805, $E=\beta^2$, $G=1$, $E_1 = 2\beta\beta_1$,

$$\therefore 2\beta^2 (p'q'' - q'p'') - 2\beta\beta_1 q'p'^2 = 0. \quad (1)$$

Since the line of striction makes a constant angle with the generators $dq_0 = C\beta dp$, $\therefore \log q_0' - \log p' = \log C + \log \beta$,

$$\therefore q_0''/q_0' - p''/p' = p'\beta_1/\beta, \ \therefore p'q_0'' - q'_0 p'' = q_0' p'^2 \beta_1/\beta,$$

which compared with (1), shews that such a line of striction is a geodesic line.

(5) See first paragraph of Art. 833.

(6) The tangent plane at any point of a generating line Aa depends only on the position of a consecutive generator Bb; hence a twist about Bb in the deformation does not change the relative positions of the tangent planes.

(7) In the differential equation of Art. 805, shew that $E=\sin^2 q$, $G=1$, $F=0$, and that the equation becomes, if p be made the independent variable instead of t,

$$2 \sin^2 q\, d^2q/(dp)^2 - 2 \sin^3 q \cos q - 4 \sin q \cos q\, (dq/dp)^2 = 0,$$

$$\therefore 2 \frac{dq}{dp} \frac{d^2q}{dp^2} - 4 \cot q \left(\frac{dq}{dp}\right)^3 = 2 \sin q \cos q \frac{dq}{dp};$$

$$\therefore \operatorname{cosec}^4 q\, (dq/dp)^2 = -\operatorname{cosec}^2 q + \operatorname{cosec}^2 \alpha,$$

$$\therefore p - \gamma = -\int d \cot q / \sqrt{(\cot^2\alpha - \cot^2 q)} = \cos^{-1} (\cot q \tan \alpha),$$

$$\therefore \cos (p - \gamma) = \cot q \tan \alpha.$$

Let P be the pole, and PB the first meridian, and let the longitude and co-latitude of Q be $\angle BPQ = p$ and $PQ=q$, those of A, $\angle BPA = \gamma$, $PA = \alpha$; $\therefore$, in the spherical triangle PAQ, $\cos APQ = \cot PQ \tan PA$, $\therefore$ QA is a great circle perpendicular to PA.

LVIII.

(1) Since α', β' are functions of α and β,

$$\lambda \{(d\alpha)^2 + (d\beta)^2\} = \lambda' \left\{\left(\frac{d\alpha'}{d\alpha} d\alpha + \frac{d\alpha'}{d\beta} d\beta\right)^2 + \left(\frac{d\beta'}{d\alpha} d\alpha + \frac{d\beta'}{d\beta} d\beta\right)^2\right\},$$

which is true for an infinite number of values of $d\alpha : d\beta$;

$$\therefore \left(\frac{d\alpha'}{d\alpha}\right)^2 + \left(\frac{d\beta'}{d\alpha}\right)^2 = \left(\frac{d\alpha'}{d\beta}\right)^2 + \left(\frac{d\beta'}{d\beta}\right)^2, \text{ and } \frac{d\alpha'}{d\alpha}\frac{d\alpha'}{d\beta} + \frac{d\beta'}{d\alpha}\frac{d\beta'}{d\beta} = 0.$$

Let $\dfrac{d\beta'}{d\alpha} = \omega \dfrac{d\alpha'}{d\alpha}$, $\therefore \omega \dfrac{d\beta'}{d\beta} = -\dfrac{d\alpha'}{d\beta}$, and $\dfrac{d\alpha'}{d\beta} = \mp \omega \dfrac{d\alpha'}{d\alpha} = \mp \dfrac{d\beta'}{d\alpha}$,

hence $\dfrac{d(\alpha'+\beta'i)}{d\beta} = \pm i \dfrac{d(\alpha'+\beta'i)}{d\alpha}$; $\therefore\ \alpha'+\beta'i = f(\alpha+\beta i)$.*

This theorem does not depend on the network upon the surface being of squares, it may be of any similar rectangles.

(2) Take p, q elliptic coordinates of any point on the ellipsoid, H, K are the reciprocals of the principal radii of curvature, corresponding to p and q constant.

By Art. 789, $P^2 = p(p-q)/4(a+p)(b+p)(c+p)$,
by Arts. 291, 720, $K = \sqrt{(abc)}/p^{\frac{3}{2}}q^{\frac{1}{2}}$, $H = \sqrt{(abc)}/p^{\frac{1}{2}}q^{\frac{3}{2}}$, and, by Art. 810, $T=0$. The equation (7), Art. 815, becomes $K_2P + (K-H)P_2 = 0$ and is satisfied by the above values of H, K, and P.

(3) In this system of coordinates, the position of any point P in the hyperboloid is given by taking two fixed generating lines of opposite systems OM, ON, and drawing through P two generators PM, PN; the position of P is given by the values $OM = p$, $ON = q$.

The equation of a plane curve must be such that for a given value of p there is only one value of q and *vice versâ*; the equation must therefore be of the form proposed. It should be observed, however, that the equation appears in this form only for a particular method of determining the generating lines, any single valued functions of p and q might be substituted for p and q respectively. Thus, as in Art. 214, if

$x = a\cos(p+q)\sec(p-q)$, $y = b\sin(p+q)\sec(p-q)$, $z = c\tan(p-q)$,
p and q constant would fix two generating lines of opposite systems, and, for any plane curve, if $Ax+By+Cz+D=0$,

$$A'\tan p\tan q + B'\tan p + C'\tan q + D' = 0.$$

(4) Let the equation of the hyperboloid be $(x^2+y^2)/a^2 - z^2/c^2 = 1$, where $a = c\tan\beta$, and let q be the length of a generating line between the points (x, y, z) and $(a\cos\alpha, a\sin\alpha, 0)$, so that $z = q\cos\beta$, $x = a\cos\alpha - q\sin\beta\sin\alpha$, $y = a\sin\alpha + q\sin\beta\cos\alpha$, whence

$$(ds)^2 = (dq)^2 + 2a\sin\beta\, d\alpha\, dq + (q^2\sin^2\beta + a^2)(d\alpha)^2,$$

which can be expressed in several different forms, viz.

i. $(ds)^2 = (dq\sin\beta + a\,d\alpha)^2 + \{(dq)^2 + (q\tan\beta\, d\alpha)^2\}\cos^2\beta$,

which, if $a\,d\alpha = c\,dp$, gives the element of an arc traced on the surface given by $x = q\cos\beta\cos p$, $y = q\cos\beta\sin p$, $z = cp + q\sin\beta$, and constructed as follows: in Oz take $OA = cp = a\alpha$; in a plane through Oz inclined to the plane zx at an angle p, draw a straight line AP, making with Oz an angle $\frac{1}{2}\pi - \beta$, then AP generates a surface defined as a helicoidal surface in Art. 837. Hence the hyperboloid can be deformed so that the arc $a\alpha$ of the principal

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* Monge ed. *Liouville*, p. 572.

circle is bent into a straight line, and the generating lines of the hyperboloid become those of the surface.

ii. $(ds)^2 = (dq + adp)^2 + (q^2 + a^2 \cot^2\beta)(dp)^2$,

where $p = \alpha \sin\beta$, in this form ds is an element of an arc described on a surface for which

$$dx = -(dq + a\,dp)\sin p - q\cos p\,dp, \quad\text{or}\quad x = -q\sin p + a\cos p,$$
$$dy = -(dq + a\,dp)\cos p + q\sin p\,dp, \quad\text{or}\quad y = -q\cos p - a\sin p,$$
$$dz = a\cot\beta\,dp, \quad\text{or}\quad z = a\cot\beta\,p = a\alpha\cos\beta,$$

thus the arc $a\alpha$ of the circle is bent on the arc of the helix whose pitch is $\frac{1}{2}\pi - \beta$.

iii. $(ds)^2 = (q^2\sin^2\beta + a^2)\{d\alpha + a\sin\beta\,dq/(q^2\sin^2\beta + a^2)\}^2$
$+ (dq)^2(q^2\sin^2\beta + a^2\cos^2\beta)/(q^2\sin^2\beta + a^2)$,

or, since $a = c\tan\beta$, $= (q^2+c^2\sec^2\beta)(dp)^2 + (dq)^2(q^2+c^2)/(q^2+c^2\sec^2\beta)$,

where $dp = \sin\beta\,\{d\alpha + c\sec\beta\,dq/(q^2 + c^2\sec^2\beta)\}$;

let $q^2 + c^2\sec^2\beta = r^2$;

then $(ds)^2 = r^2dp^2 + (dr)^2\{1 + c^2/(r^2 - c^2\sec^2\beta)\}$
$= (rdp)^2 + (dr)^2 + (dx)^2$, where $r = c\sec\beta\cosh(x/c)$,
and $p = \sin\beta\,\{\alpha + \tan^{-1}(q\cos\beta/c)\}$,

shewing the applicability to a surface of revolution,

$$\sqrt{(y^2 + z^2)} = c\sec\beta\cosh(x/c).$$

$q = 0$ corresponds to the principal circular section of the hyperboloid, and also to that of the surface to which it is applied, on which it extends over an angle $2\pi\sin\beta$.

(5) See fig. p. 350. A surface of revolution into which the sphere may be deformed is that generated by double $A'D'E'$ not unlike the arc of a circle, OE' being the half-chord and OA' the versed-sine, $OE' > OA$ and $OA' < OA$.

Figure p. 351. A zone of the sphere may be deformed into a portion of a surface of revolution generated by double $A'F'$, which resembles a semi-ellipse, revolving about OE, the semi-axis being less than OE and the greatest radius OA' greater than OA, the latitudes of the bounding small circles of the zone being not greater than $\sin^{-1}(OA/OA')$.*

(6) Let the surface be referred to the tangent plane and principal normal planes as coordinate planes; near the origin O its equation is $2z = ax^2 + by^2$. Let s be the small arc of a geodesic through O, whose tangent at O makes an angle α with Ox, then, denoting dx/ds by x', &c., $x' = \cos\alpha$ and $y' = \sin\alpha$, when $s = 0$;

$$z' = axx' + byy' \quad\text{and}\quad z'' = a(xx'' + x'^2) + b(yy'' + y'^2);$$

hence, when $s = 0$, $z' = 0$, $z'' = a\cos^2\alpha + b\sin^2\alpha$.

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* Cayley, *Mess. of Math.*, vol. VI. p. 88.

By the equations of a geodesic

$$x'' + axz'' = 0, \ \therefore\ x''' + axz''' + ax'z'' = 0,$$

hence, when $s = 0$, $x'' = 0$, $x''' = -a\cos\alpha\,(a\cos^2\alpha + b\sin^2\alpha)$.

Hence, if (ξ, η, ζ) be the extremity of the radius ρ of the geodesic circle, along which s is measured, by Maclaurin's theorem, neglecting terms in ρ^4,

$$\xi = \rho\cos\alpha - \tfrac{1}{6}\rho^3\,(a^2\cos^3\alpha + ab\cos\alpha\sin^2\alpha),$$

similarly $\eta = \rho\sin\alpha - \tfrac{1}{6}\rho^3\,(b^2\sin^3\alpha + ab\sin\alpha\cos^2\alpha)$,

and $\zeta = \tfrac{1}{2}\rho^2\,(a\cos^2\alpha + b\sin^2\alpha)$.

Let $d\sigma$ be a small arc of the geodesic circle, the extremities corresponding to α and $\alpha + d\alpha$,

$$d\xi/d\alpha = -\rho\sin\alpha + \tfrac{1}{6}\rho^3\,\{3a^2\cos^2\alpha\sin\alpha + ab\,(\sin^3\alpha - 2\sin\alpha\cos^2\alpha)\},$$
$$d\eta/d\alpha = \rho\cos\alpha - \tfrac{1}{6}\rho^3\,\{3b^2\sin^2\alpha\cos\alpha + ab\,(\cos^3\alpha - 2\cos\alpha\sin^2\alpha)\},$$
$$d\zeta/d\alpha = \rho^2\,(b - a)\sin\alpha\cos\alpha;$$
$$\therefore\ (d\sigma/d\alpha)^2 = \rho^2 - \tfrac{1}{3}\rho^4 ab\,(\sin^4\alpha + \cos^4\alpha - 4\sin^2\alpha\cos^2\alpha) - 2\rho^4 ab\sin^2\alpha\cos^2\alpha$$
$$= \rho^2\,(1 - \tfrac{1}{3}\rho^2 ab),$$

and the perimeter of the circle is $2\pi\rho - \tfrac{1}{3}\pi\rho^3 ab$.*

On deformation of the surface, the circle remains a geodesic circle, with the same radius, therefore the specific curvature at O, which is ab, remains unaltered.

The area of the geodesic circle is

$$\int_0^{2\pi}\!\!\int_0^{\rho} ds\,d\alpha\,s\,(1 - \tfrac{1}{6}s^2 ab) = \pi\rho^2\,(1 - \tfrac{1}{12}\rho^2 ab).$$

(7) By Art. 832 the change of the angle of contingence of the normal section perpendicular to a given generating line is γdp, the angle between consecutive shortest distances, which distances are perpendicular to consecutive facets of the director cone.

LIX.

(1) Fig 7. Let $aPQb$ be one of the curves cutting orthogonally the generating lines of one system, AMN the particular curve which passes through A where $\theta = 0$, $\phi = 0$, and let PM, QN be two consecutive generators intersecting the principal circular section AB in P', Q', the angles AOP' and AOQ' being p and $p + dp$; and since the projection of $P'P$ on the plane AOP' is the tangent $P'T$ at P', and β is the angle between any generating line and the axis Oz, $P'T = z\tan\beta = a\tan\phi$, $\therefore\ TOP' = \phi$ and $p + \phi = \theta$.

Let $MP = q = NQ$ and $MP' = q'$, then $dq' = NQ' - MP' = adp\sin\beta$,

$$\therefore\ q' = ap\sin\beta;\ \text{also}\ (q - q')\cos\beta = z = a\cot\beta\tan\phi,$$
$$\therefore\ \tan\phi = (q/a - p\sin\beta)\sin\beta.$$
$$\therefore\ \theta - \phi + \operatorname{cosec}^2\beta\tan\phi = \operatorname{cosec}\beta\, q/a.$$

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* See *Puiseux* quoted in Monge ed. *Liouville*, p. 586.

(2) Using the hyperboloid of the last problem,

$$PQ = ap \sin\beta,\ PQ' = a(p + 2\pi)\sin\beta,\ \&c.;$$
$$\therefore\ QQ' = Q'Q'' = \ldots = 2\pi a \sin\beta,$$

which is independent of the position of the generating line intersected by the curve AQ.

$$\text{At } Q,\ z = -PQ\cos\beta = -ap\sin\beta\cos\beta,$$
$$\therefore\ x = a\cos p + ap\sin^2\beta\sin p,\ y = a\sin p - ap\sin^2\beta\cos p,$$
$$\therefore\ (ds)^2 = a^2(dp)^2(\cos^4\beta + p^2\sin^4\beta) + a^2(dp)^2\sin^2\beta\cos^2\beta$$
$$= a^2(dp)^2(\cos^2\beta + p^2\sin^4\beta),$$
$$\text{and } \cot\psi = p\sin\beta\tan\beta,\ \therefore\ ds = a\cot^2\beta\operatorname{cosec}\psi\, d\cot\psi;$$
$$\therefore\ s = \tfrac{1}{2}a\cot^2\beta\{\cot\psi\operatorname{cosec}\psi + \log(\operatorname{cosec}\psi + \cot\psi)\}.$$

(3) Let the conicoid be $x^2/a + y^2/b + z^2/c = 1$, (1), and the consecutive confocal $x^2/(a+k) + \ldots = 1$; if ϖ, ϖ' be the perpendiculars from the centre on parallel tangent planes $\varpi'^2 - \varpi^2 = k$, hence, the distance from the point of contact with (1) to the consecutive confocal is ultimately $\varpi' - \varpi = k/(\varpi' + \varpi)$, which varies as ϖ^{-1}, ultimately.

Let the curves (p), (p'), (q), (q'), determine the lines of curvature which are the sides of the quadrilateral; for the point whose elliptic coordinates are p, q, $\varpi^2 pq = abc$, and $pq : pq' = p'q : p'q'$, whence the theorem.

(4) Use the equation of the wave surface

$$ax^2/(\rho^2 - a) + by^2/(\rho^2 - b) + cz^2/(\rho^2 - c) = 0,\ (1),\quad \rho^2 = x^2 + y^2 + z^2.$$

Let the elliptic coordinates p, q, r be taken belonging to the conicoid $x^2/-a + y^2/-b + z^2/-c = 1$, being the roots of

$$(k-a)(k-b)(k-c) - x^2(k-b)(k-c) - \ldots = 0,$$
$$\text{so that } p + q + r = a + b + c + \rho^2;\quad (2)$$

arranging according to powers of $k - a$,

$$(k-a)^3 + P(k-a)^2 + Q(k-a) - x^2(a-b)(a-c) = 0,$$
$$\therefore\ x^2(a-b)(a-c) = -(a-p)(a-q)(a-r);$$
$$\text{and so, by (1), } \frac{a(a-p)(a-q)(a-r)}{(\rho^2-a)(a-b)(a-c)} + \ldots = 0.\quad (3)$$

$$\text{Let } \frac{t(t-p)(t-q)(t-r)}{(\rho^2-t)(t-a)(t-b)(t-c)} \equiv \frac{A}{t-a} + \frac{B}{t-b} + \frac{C}{t-c} + \frac{D}{\rho^2-t} - 1,$$
$$A = \frac{a(a-p)(a-q)(a-r)}{(\rho^2-a)(a-b)(a-c)},\quad D = \frac{\rho^2(\rho^2-p)(\rho^2-q)(\rho^2-r)}{(\rho^2-a)(\rho^2-b)(\rho^2-c)},$$

clearing of fractions and equating the coefficients of t^3,

$$A + B + C - D = p + q + r - a - b - c - \rho^2 = 0,\ \text{by (2)};$$
$$\therefore\ D = A + B + C = 0,\ \text{by (3)};$$
$$\therefore\ (p-\rho^2)(q-\rho^2)(r-\rho^2) = 0,\ \text{the result required.}$$

See Cayley on this equation, *Mess. of Math.* vol. VIII. p. 191.

(5) $dP/dt = ma\{(n+1)t^n + (n-1)t^{n-2}\}$, $dq/dt = at^{n-2}(t^2+1)$,
$$P\,dt/dq = mt,\quad dP/dq = m\{n+1-2/(t^2+1)\},$$
$$\therefore\ P\frac{d^2P}{dq^2} = 4m^2t^2/(t^2+1)^2, \text{ and } P^3\frac{d^2P}{dq^2} = 4m^4a^2t^{2n},$$
$$\frac{d}{dt}\log\left(P^3\frac{d^2P}{dq^2}\right) = \frac{2n}{t},\quad P\frac{d}{dq}\log\left(P^3\frac{d^2P}{dq^2}\right) = 2mn.*$$

(6) By Art. 841, since in a surface of revolution of which the curves (p) are meridians P is a function of q only, $\phi_2 = 0$; $\therefore\ \phi \equiv 2\theta$ is a function of p only, which proves i., since θ is constant for the same meridian. Also $\phi_1 = mn$, $\therefore\ \phi \equiv 2\theta = mn(p+p_0)$; $\therefore\ \theta - \theta' \propto p - p'$ for the same line of curvature, which proves ii.

(7) Let the equation of the surface generated by the revolution of the hypocycloid about the axis of x be $r^{\frac{2}{3}} + x^{\frac{2}{3}} = c^{\frac{2}{3}}$, where $r^2 = y^2 + z^2$. Take q for the arc of a meridian measured from a cusp A in the axis of x to a point P; ds an element PQ of a curve drawn through P is given by $(ds)^2 = (dq)^2 + r^2(dp)^2$, where dp is the angle between the meridian planes AP, AQ; and $r^2 = \frac{8}{27}q^3/c$.
$$\text{Let } r = r'\sec\alpha,\quad p = p'\cos\alpha,$$
$$\text{then } (ds)^2 = (dq)^2 + r^2(dp)^2 = (dq)^2 + r'^2(dp')^2;$$
hence, since $r'^2 = \frac{8}{27}q^3/c'$, where $c' = c\sec^2\alpha$, the given surface is applicable to another surface of revolution, whose equation is $r'^{\frac{2}{3}} + x^{\frac{2}{3}} = c'^{\frac{2}{3}}$, namely, a surface generated by a hypocycloid similar to the former whose linear dimensions are greater than those of the former in the ratio $\sec^2\alpha : 1$.

The arc of the generating hypocycloid between two cusps extends, when bent along that of the new surface, only to a point where $r' = c\cos\alpha$, and $dr'/dq = (r'/c')^{\frac{1}{3}} = \cos\alpha$. (1)

If the two halves of the first surface be bent on the corresponding sheets of the second surface, and the unoccupied portions be removed, the occupied portions can be placed together, so as to become a surface with an edge not cuspidal but at which the sheets intersect at an angle 2α, by (1).

(8) Using the figure and notation of Art. 831, $\tan I = (q-q_0)/\beta$, in which as P moves along Aa, β and q_0 are unaltered,
$$\therefore\ \sec^2 I\,dI/dt = V/\beta,\quad \therefore\ dI/dt = V\cos^2 I/\beta,$$
and $-\cos^4 I/\beta^2$ is the specific curvature at $P = -(R_1R_2)^{-1}$.

LX.

(1) i. Let p, q; p, q'; p', q'; p', q be the elliptic coordinates of the angular points A, B, C, D of the quadrilateral.

* Bour, *Jour. de L'Ec.* Pol. I. Cah. 39, p. 99.

$$p+q=OA^2-a-b-c,\ p'+q'=OC^2-a-b-c,$$

$$\therefore\ OA^2+OC^2=p+q+p'+q'+2(a+b+c)=OB^2+OD^2.\ (1)$$

ii. Let (l, m, n) and (l', m', n') be the directions of OA and OC,

$\therefore$, by Art. 287, $l^2.OA^2=a(a+p)(a+q)/(a-b)(a-c)$,

$$l'^2.OC^2=a(a+p')(a+q')/(a-b)(a-c),$$

$$\therefore\ ll'.OA.OC=\sqrt{\{(a+p)(a+q)(a+p')(a+q')\}}\,a/(a-b)(a-c),$$

and if (λ, μ, ν) and (λ', μ', ν') be the directions of OB and OD,

$$ll'.OA.OC=\lambda\lambda'.OB.OD.$$

Thus $OA.OC(ll'+mm'+nn')=OB.OD(\lambda\lambda'+\mu\mu'+\nu\nu')$,

or $OA.OC\cos AOC=OB.OD\cos BOD$;

$\therefore\ OA^2+OC^2-AC^2=OB^2+OD^2-BD^2$, and, by (1), $AC=BD$.

iii. Let $x^2/(a+r)+\ldots=1$ be the confocal ellipsoid,

then $p+q+r=OA'^2-a-b-c$;

$\therefore\ OA'^2-OA^2=r$, similarly for B', C', D'.

(2) Let the conicoids (1), (2), and (3) be respectively an ellipsoid, a hyperboloid of one sheet, and of two sheets, so that $p>q>r$. By Art. 296, $p-q$, $p-r$ are the squares of the semi-axes of the central section of the ellipsoid by a plane parallel to the tangent plane at the point of intersection of the three conicoids. Hence, by Art. 720, $\rho_q/\rho_r=(p-r)/(p-q)$; similarly, since the two centres of principal curvature of the hyperboloid of one sheet are on opposite sides of the tangent plane,

$$\sigma_r/\sigma_p=-(p-q)/(q-r);\quad \tau_p/\tau_q=(q-r)/(p-r);$$

$$\therefore\ \rho_q\sigma_r\tau_p+\rho_r\sigma_p\tau_q=0,\quad \sigma_r/\sigma_p+\tau_q/\tau_p=1,\ \&c.$$

(3) By Art. 502, $p^{-2}=u^2+(du/d\theta)^2+\operatorname{cosec}^2\theta\,(du/d\phi)^2$,

$$\therefore\ \sec^2\psi=r^2/p^2=1+(d\log r/d\theta)^2+\operatorname{cosec}^2\theta\,(d\log r/d\phi)^2;$$

$$\therefore\ \tan^2\psi=P^2+Q^2.$$

In the case of the ellipsoid $ax^2+by^2+cz^2=1$,

$$r^{-2}=\sin^2\theta\,(a\cos^2\phi+b\sin^2\phi)+c\cos^2\theta,$$

$$P=d\log r/d\theta=-r^2\sin\theta\cos\theta\,(a\cos^2\phi+b\sin^2\phi-c),$$

$$Q=\operatorname{cosec}\theta\,d\log r/d\phi=-r^2\sin\theta\sin\phi\cos\phi\,(b-a),$$

$$(1+P^2)\,r^{-4}=\sin^2\theta\,(a\cos^2\phi+b\sin^2\phi)^2+c^2\cos^2\theta,$$

$$(1+P^2+Q^2)\,r^{-4}=\sin^2\theta\,(a^2\cos^2\phi+b^2\sin^2\phi)+c^2\cos^2\theta;$$

$$\therefore\ 1+P^2+Q^2=r^2(a^2x^2+b^2y^2+c^2z^2)=r^2/p^2=\sec^2\psi.$$

(4) Since $(q^2+1)(1+pq)+(q^2-1)(1-pq)=2q^2+2pq$,

$$\therefore\ (q^2+1)\,x/a+(q^2-1)\,z/c=2q,$$

hence all the points of the striating line for which q is constant lie in a plane; similarly for p; hence the striating lines are generating

lines, and the lines of curvature bisect the angles between the lines (p) and (q) through which they pass,

$$\therefore\ E(dp)^2 - G(dq)^2 = 0, \text{ Art. 796.}$$

Shew that $(ds)^2(p+q)^4 = a^2\{(q^2-1)\,dp + (p^2-1)\,dq\}^2$
$+ 4b^2(q\,dp - p\,dq)^2 + c^2\{(q^2+1)\,dp + (p^2+1)\,dq\}^2$,

hence, that $E(p+q)^4/(a^2+c^2) = q^4 - 2Aq^2 + 1$,
$G(p+q)^4/(a^2+c^2) = p^4 - 2Ap^2 + 1$.

(5) By Art. 831, the specific curvature at any point (p, q) is $-\beta^2/\{(q-q_0)^2+\beta^2\}^2$, which is zero where $q-q_0$ is finite, but is $-\beta^{-2}$ at the point where the generating line meets the line of striction.

The explanation of the discontinuity is given by making $q-q_0=x$ and the specific curvature y, tracing the curve, and observing the form as β gradually diminishes. Fig. 8 represents the two forms when $\beta=1$ and $\frac{1}{2}$. In the general case the point of inflexion is where $x=\beta/\sqrt{5}$, and the radius of curvature where $x=0$, $y=-\beta^{-2}$ is $\frac{1}{4}\beta^4$.

The curves corresponding to β and β' intersect where $x^2=\beta\beta'$.

(6) Fig. p. 347. The equations of a generating line Aa through the point $(a\cos\theta, a\sin\theta, 0)$ are, if $c = a\tan\alpha$,

$$x = (a - z\cot\alpha)\cos\theta, \quad y = (a + z\cot\alpha)\sin\theta.$$

Let (x, y, z) and $(x+\delta x, y+\delta y, z+\delta z)$ be the points A, B' in which the line of shortest distance meets Aa and the consecutive generator Bb; since AB' is perpendicular to Aa and Bb,

$$-\delta x\cos\theta + \delta y\sin\theta + \delta z\tan\alpha = 0, \quad (1)$$

$$\text{and } \delta x\sin\theta + \delta y\cos\theta = 0; \quad (2)$$

$$\text{also } \delta x = -\delta z\cot\alpha\cos\theta - (a - z\cot\alpha)\sin\theta\,d\theta,$$

$$\text{and } \delta y = \delta z\cot\alpha\sin\theta + (a + z\cot\alpha)\cos\theta\,d\theta,$$

by (1), $\delta z = -a\sin\alpha\cos\alpha\sin 2\theta\,d\theta$, and by (2), $z = -a\tan\alpha\cos 2\theta$;

$$\therefore\ x = 2a\cos^3\theta, \quad y = 2a\sin^3\theta, \text{ are coordinates of } A.$$

To find the four elements of the scroll, viz. AB', $B'B$ and the angles, $d\psi$ between Aa, Bb, and $d\phi$ between AB', BC',

$$\delta x/\cos\theta = \delta y/-\sin\theta = \delta z\tan\alpha = -a\sin^2\alpha\sin 2\theta\,d\theta;$$

$$\therefore\ AB'^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2 = a^2\sin^2\alpha\sin^2 2\theta\,(d\theta)^2,$$

$$AB^2 = \{(-3\sin 2\theta\cos\theta)^2 + (3\sin 2\theta\sin\theta)^2 + (-2\tan\alpha\sin 2\theta)^2\}\,a^2(d\theta)^2,$$

$$\therefore\ B'B^2 = (9 + 4\tan^2\alpha - \sin^2\alpha)\,a^2\sin^2 2\theta\,(d\theta)^2,$$

whence $B'B = (3 + 2\tan^2\alpha)\,a\cos\alpha\sin 2\theta\,d\theta = d\sigma$, suppose.

The direction-cosines of Aa are $-\cos\theta\cos\alpha$, $\sin\theta\cos\alpha$, and $\sin\alpha$,

$$\therefore\ \cos d\psi = \cos^2\alpha\,\{\cos\theta\cos(\theta+d\theta) + \sin\theta\sin(\theta+d\theta)\} + \sin^2\alpha$$
$$= \cos^2\alpha\cos d\theta + \sin^2\alpha,$$

$$\therefore\ d\psi = \cos\alpha\,d\theta; \quad \text{similarly } d\phi = \sin\alpha\,d\theta.$$

In the deformation proposed, all the generating lines are parallel to a fixed plane, to which the lines of shortest distances are perpendicular, and they are all tangents to a cylindrical surface whose base is the limit of the polygon of which the sides are projections of such lines as $B'B$ on the fixed plane; hence if ψ be the inclination of BB' to a fixed line in the plane, the intrinsic equation of the base of the cylinder is $d\sigma/d\psi = (3 + 2\tan^2\alpha)\, a \sin(2\sec\alpha\, \psi)$, which is that of a hypocycloid; the radii R, r of the fixed and moving circles are given by

$$\tfrac{1}{2}\cos\alpha = 1 - 2r/R, \text{ and } 4r(R-r)/(R-2r) = (3 + 2\tan^2\alpha)\, a.$$

The locus of the points of contact of the generators of the deformed scroll with the cylindrical surface is a curve, the tangent of whose inclination to the base is the limit of $B'B/AB'$, which is constant.

(7) The element ds of a curve drawn in any direction on the sphere is unaltered in length when the sphere is deformed, hence for all values of $dp : dq$

$$\cos^2 q\,(dp)^2 + (dq)^2$$
$$= (1+\delta\alpha)^2 \left[\cos^2(q+\delta q)\{d(p+\delta p)\}^2 + \{d(q+\delta q)\}^2\right] + (d\delta\alpha)^2,$$

and retaining only the first powers of the increments δp, δq, $\delta\alpha$,

$$(\cos^2 q\,\delta\alpha - \sin q\cos q\,\delta q)(dp)^2 + \cos^2 q\, d\,\delta p\, dp + \delta\alpha\,(dq)^2 + d\,\delta q\, dq = 0;$$

$$\text{observe that } d\,\delta p \equiv dp\frac{d\,\delta p}{dp} + dq\frac{d\,\delta p}{dq}, \text{ \&c.,}$$

and equate the coefficients of $(dp)^2$, $dp\,dq$ and $(dq)^2$ to zero,

$$\text{whence } \cos^2 q\,\delta\alpha - \sin q\cos q\,\delta q + \cos^2 q\, d\,\delta p/dp = 0,$$
$$\cos^2 q\, d\,\delta p/dq + d\,\delta q/dp = 0, \text{ and } \delta\alpha + d\,\delta q/dq = 0, \quad (1)$$
$$\therefore \sec q\, d\,\delta q/dq + \tan q\sec q\,\delta q - \sec q\, d\,\delta p/dp = 0,$$
$$\text{or } d(\sec q\,\delta q)/dq - \sec q\, d\,\delta p/dp = 0,$$
$$\text{also } \cos q\, d\,\delta p/dq + d(\sec q\,\delta q)/dp = 0, \text{ and } du/dq = \sec q,$$
$$\therefore d(\sec q\,\delta q)/du - d\,\delta p/dp = 0, \quad (2)$$
$$d(\sec q\,\delta q)/dp + d\,\delta p/du = 0;$$
$$\therefore \frac{d^2\delta p}{du^2} + \frac{d^2\delta p}{dp^2} = 0.$$

The most general real value of δp is, $f(z)$ and $\phi(z)$ being real functions, $f(p+iu) + f(p-iu) + i\{\phi(p+iu) - \phi(p-iu)\}$.

$$\text{Let } f(z) = C\cos sz \text{ and } \phi(z) = D\sin sz,$$
$$\delta p = 2C\cos sp\cos siu + 2iD\cos sp\sin siu$$
$$= C\cos sp\,(e^{-su} + e^{su}) + D\cos sp\,(e^{-su} - e^{su}),$$
$$\text{and } e^u = \{1 + \cos(\tfrac{1}{2}\pi - q)\}/\sin(\tfrac{1}{2}\pi - q) = \cot(\tfrac{1}{4}\pi - \tfrac{1}{2}q),$$

therefore, writing A for $C+D$ and B for $C-D$,

$$\delta p = \cos sp\,\{A\tan^s(\tfrac{1}{4}\pi - \tfrac{1}{2}q) + B\cot^s(\tfrac{1}{4}\pi - \tfrac{1}{2}q)\}.$$

$$\text{By (2), } d(\sec q\,\delta q)/du = -s\sin sp\,(Ae^{-su}+Be^{su}),$$
$$\therefore\ \sec q\,\delta q = \sin sp\,(Ae^{-su}-Be^{su}).$$
$$\text{By (1), } \delta\alpha = -d\,\delta q/dq = \sin sp\sin q(Ae^{-su}-Be^{su})$$
$$+\sin sp\,(Ase^{-su}+Bse^{su}).$$

LXI.

(1) The equations of the generating circle are
$$x=\alpha,\ y^2+z^2=\beta y+\gamma z,$$
and the functional equation of the surface is $y^2+z^2=yf(x)+z\phi(x)$;
$$\therefore\ y+z^2/y=f(x)+\phi(x)\,z/y,$$
$$1+2qz/y-z^2/y^2=\phi(x)\,(yq-z)/y^2.$$
Differentiate, with respect to y, the equation
$$\log(y^2-z^2+2qyz)-\log(yq-z)=\log\phi(x).$$

(2) The equations of the generating line are $y=\alpha x$, $x=\beta z+\gamma$, where $(1+\alpha^2)\gamma^2=a^2$, $\therefore\ \beta z=x-ax/\sqrt{(x^2+y^2)}$, and β is an arbitrary function of α or y/x; multiply by $\sqrt{(x^2+y^2)}/x$, then $zf(y/x)=\sqrt{(x^2+y^2)}-a$; shew that $(px+qy)/z=\sqrt{(x^2+y^2)}/\{\sqrt{(x^2+y^2)}-a\}$.

(3) The equation of the conoid, having Oz for axis, must be of the form $y(\alpha z^2+2\beta z+\gamma)=x(\alpha' z^2+2\beta' z+\gamma')$, Art. 854, any plane $y=mx$ contains the axis and two generating lines corresponding to $z=z_1$ and $z=z_2$, z_1, z_2 being roots of
$$(m\alpha-\alpha')\,z^2+2\,(m\beta-\beta')\,z+m\gamma-\gamma'=0.$$

(4) The equation can be written in the form
$$z^2(z-x+z-y)^2+2z(a-z)\{z-x-(z-y)\}^2-2a^2(z-x)(z-y)=0;$$
$\therefore\ (z-x)/(z-y)=f(z)$; the required result follows, Art. 854.

(5) The middle point of a chord inclined to Ox at an angle α is $(a\cos^2\alpha,\ a\cos\alpha\sin\alpha,\ 0)$, and the equation of the corresponding sphere is $x^2+y^2+z^2=2a\,(x\cos^2\alpha+y\cos\alpha\sin\alpha)$,
$$\text{or } x^2+y^2+z^2-ax=ax\cos 2\alpha+ay\sin 2\alpha,$$
for the envelope $-x\sin 2\alpha+y\cos 2\alpha=0$. Eliminate α.

(6) For the envelope, $x\,dl+y\,dm+z\,dn=0$, $l\,dl+m\,dm+n\,dn=0$, and $\lambda\,dl+\mu\,dm+\nu\,dn=0$; if $x+Al+B\lambda=0$ and $y+Am+B\mu=0$, then $z+An+B\nu=0$;
$$\therefore\ lx+my+nz+A\,(l^2+m^2+n^2)=0,\ \text{or } a+A=0;$$
$$\therefore\ x^2+y^2+z^2-a^2=-B\,(\lambda x+\mu y+\nu z),$$
$$\lambda x+\mu y+\nu z=-B\,(\lambda^2+\mu^2+\nu^2).\ \text{ Eliminate } B.$$

(7) Let $y^2/b+z^2/c=x$ be the equation of the paraboloid; the squares of the semi-axes of the section by the plane $x=\alpha$ are $b\alpha$, $c\alpha$, and the equation of the ellipsoid of which this is a principal section

is $(x-\alpha)^2/a+y^2/b+z^2/c=\alpha$, if each of the series be similar to $x^2/a+y^2/b+z^2/c=1$. We have for the envelope $-2(x-\alpha)/a=1$, hence the equation is $y^2/b+z^2/c-x=\frac{1}{4}a$.

(8) Shew, as in Art. 240, that the area of the section of a hyperboloid of one sheet, whose equation is $ax^2+by^2+cz^2=\lambda$, c being negative, is equal to $(1+p^2/\varpi^2\lambda)\,\pi\lambda\,(-abc)^{-\frac{1}{2}}/\varpi$, where $\varpi^2=-l^2/a-m^2/b-n^2/c$.

Hence, when $\lambda=0$, in which case the hyperboloid becomes a cone, the area of the section of the cone by the plane $lx+my+nz=p$ is $\pi(-abc)^{-\frac{1}{2}}p^2/\varpi^3$; and, since the volume cut off by the plane is constant, $(p/\varpi)^3$ is constant. Thus the equation of the cutting plane is $lx+my+nz=p=C\sqrt{(-l^2/a-m^2/b-n^2/c)}$, where C is constant, the plane is therefore a tangent plane to the hyperboloid $ax^2+by^2+cz^2=-C^2$.

LXII.

(1) Shew that the equations of the generating lines can be put in the form $mx=\alpha z+\beta c$, $y=\beta z+\alpha c$, (1)

$$\therefore\ mzx-cy=\alpha(z^2-c^2),\quad (2)$$

$$yz-mcx=\beta(z^2-c^2),\ (3)\quad \text{and}\quad \beta=f(-\alpha).$$

For any direction denoted by dx, dy, dz on the envelope, $Udx+Vdy+Wdz=0$, and if this be the direction of the generating line (1), $mdx=\alpha dz$, $dy=\beta dz$, $\therefore\ \alpha U+m\beta V+mW=0$, whence the corrected result, by (2) and (3).

(2) i. The equation of the tangent plane at (x, y, z) is

$$\zeta-z=p(\xi-x)+q(\eta-y),\ \therefore\ px+qy=z-k^{n+1}/z^n.$$

To integrate this equation $dx/x=dy/y=z^ndz/(z^{n+1}-k^{n+1})$;

$$\therefore\ y=\alpha x\quad\text{and}\quad (n+1)\log x+\log\beta=\log(z^{n+1}-k^{n+1}),$$

$$\therefore\ z^{n+1}-k^{n+1}=\beta x^{n+1}=x^{n+1}f(y/x).$$

ii. The intercepts by the tangent plane on Ox and Oy are

$$(px+qy-z)/p\ \text{and}\ (px+qy-z)/q,\ \therefore\ px=qy,$$

write t for y/x, then $(n+1)z^np=(n+1)x^nf(t)-x^{n-1}yf'(t)$,

and $(n+1)z^nq=x^nf'(t)$, $\therefore\ t=(n+1)f(t)/f'(t)-t$,

hence $f'(t)/f(t)=\frac{1}{2}(n+1)t^{-1}$, and $f(t)=Ct^{\frac{1}{2}(n+1)}$.

(3) The functional equation is obtained from

$$x/c\cos\theta+y/c\sin\theta=1\quad\text{and}\quad\theta=f(z).$$

For the differential equation

$$\sec\theta+(x\sec\theta\tan\theta-y\operatorname{cosec}\theta\cot\theta)\,pf'(z)=0,$$

$$\operatorname{cosec}\theta+(x\sec\theta\tan\theta-y\operatorname{cosec}\theta\cot\theta)\,qf'(z)=0;$$

$\therefore\ \cos\theta/q=\sin\theta/p=1/\sqrt{(p^2+q^2)}$ and $(x/q+y/p)\sqrt{(p^2+q^2)}=c$.

(4) Take Ox as the given line, and Oz containing the given point C where $OC=c$. The equation of one of the spheres is $x^2-2\gamma x+y^2+z^2=c^2$, and for a surface cutting the sphere orthogonally $(x-\gamma)p+yq-z=0$,

$$\text{or } (x^2+c^2-y^2-z^2)p/2x+yq-z=0,$$

$$2x\,dx/(x^2+c^2-y^2-z^2)=dy/y=dz/z;$$

$$\therefore\ z=\alpha y,\ 2x\,dx/y-(x^2+c^2)\,dy/y^2+(1+\alpha^2)\,dy=0,$$

$$\therefore\ (x^2+c^2)/y+(1+\alpha^2)y=\beta=f(\alpha);$$

the functional equation is $x^2+c^2+y^2+z^2=yf(z/y)$.

(5) The functional equation of the family of surfaces is

$$x^2/a+y^2/b+z^2/c=f(x^2+y^2+z^2),$$

$$\text{whence } (x/a+pz/c)(y+qz)-(y/b+qz/c)(x+pz)=0.$$

(6) The vertex of a cone of revolution enveloping an ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ is on the umbilical focal conic, and its coordinates are α, 0, γ, where $\alpha^2/(a^2-b^2)-\gamma^2/(b^2-c^2)=1$; the equation of the plane of contact is $\alpha x/a^2+\gamma z/c^2=1$, (1).

Hence, for the envelope, $\alpha d\alpha/(a^2-b^2)-\gamma d\gamma/(b^2-c^2)=0$, and $x\,d\alpha/a^2+z\,d\gamma/c^2=0$;

$$\therefore\ \frac{a^2\alpha}{x(a^2-b^2)}=\frac{c^2\gamma}{-z(b^2-c^2)}=\frac{\alpha^2/(a^2-b^2)-\gamma^2/(b^2-c^2)}{\alpha x/a^2+\gamma z/c^2}=1;$$

$$\therefore\text{, by (1), } x^2(a^2-b^2)/a^4-z^2(b^2-c^2)/c^4=1,$$

the dirigent cylinder of the focal conic.

This result follows also from the theorem that the focal and dirigent conics are reciprocals of each other with respect to the principal section in the plane of which they lie, Art. 346.

(7) For the envelope,

$$yz+\ldots=2a(x+\ldots) \text{ and } 2xyz=a(yz+\ldots);\quad (1)$$

$$\therefore\ 4xyz(x+\ldots)=(yz+\ldots)^2 \text{ or } 4\{(yz)^{-1}+\ldots\}=(x^{-1}+\ldots)^2;$$

$$\therefore\ x^{-2}+\ldots-2(yz)^{-1}-\ldots=0, \text{ whence } x^{-\frac{1}{2}}+y^{-\frac{1}{2}}+z^{-\frac{1}{2}}=0.$$

For the characteristic, $2a^{-1}=x^{-1}+y^{-1}+z^{-1}$, by (1),

$$=x^{-1}+y^{-1}+(x^{-\frac{1}{2}}+y^{-\frac{1}{2}})^2,\ \therefore\ xy=a\{x+y+(xy)^{\frac{1}{2}}\}.$$

(8) Shew that the equation of the cone, whose vertex is at (x, y, z) in the conicoid $ax^2+by^2+cz^2=1$, is $a\xi^2+b\eta^2+c\zeta^2=2u^2-2u+1$, where $u=ax\xi+\ldots$, hence, for the envelope $2u=1$, and $a\xi^2+\ldots=\frac{1}{2}$.

(9) Write Δ, Δ' for the two determinants in (1),

$$\text{and } \Delta=A\alpha+B\beta+C\gamma,\ \Delta'=A'\alpha+B'\beta+C'\gamma.$$

$$(1)\ (A\Delta'+A'\Delta)\,d\alpha+\ldots=0 \text{ and } \alpha\,d\alpha+\ldots=0;$$

hence, for the envelope,

$$(A\Delta'+A'\Delta)/\alpha=\ldots=\ldots=2\Delta'\Delta/(\alpha^2+\beta^2+\gamma^2)=2m;$$

$$\therefore\ (A^2+B^2+C^2)\,\Delta'+(AA'+BB'+CC')\,\Delta=2m\Delta,$$
$$\text{and }\ (A'^2+B'^2+C'^2)\,\Delta+(AA'+BB'+CC')\,\Delta'=2m\Delta',$$

the equation of the envelope is $(AA'+\ldots-2m)^2=(A^2+\ldots)(A'^2+\ldots)$, where $A=cy-bz$, &c.; and the equation reduces to

$$[\{x(bc'-b'c)+\ldots\}^2+4m(aa'+\ldots)](x^2+y^2+z^2)$$
$$=4m^2+4m(ax+\ldots)(a'x+\ldots).$$

(2) As in (1), writing Π for $\alpha x+\beta y+\gamma z$,

$$(x\Delta+A\Pi)/\alpha=\ldots=\ldots=2\Pi\Delta=2m\ \text{ and }\ Ax+By+Cz=0,$$
$$\therefore\ (A^2+B^2+C^2)\,\Pi=2m\Delta\ \text{ and }\ (x^2+y^2+z^2)\,\Delta=2m\Pi,$$

hence, the equation of the envelope is

$$(x^2+y^2+z^2)\,\{(a^2+\ldots)(x^2+\ldots)-(ax+\ldots)^2\}=4m^2.$$

LXIII.

(1) The equations of the two spheres are $r^2+2\alpha x=a^2$, and $r^2+2\beta y=c^2$, and the characteristic is a circle in the plane $\alpha x=\beta y$; hence, p and q being the same for the spheres and surfaces generated,

$$(x+\alpha)\,dx+y\,dy+z\,(p\,dx+q\,dy)=0\ \text{ and }\ \alpha dx=\beta dy;$$
$$\therefore\ (x+\alpha)\,\beta+y\alpha+z\,(p\beta+q\alpha)=0,$$
$$\therefore\ (x+zp)\,2x/(r^2-a^2)+(y+zq)\,2y/(r^2-c^2)=1.$$

The functional equation of the surface is, since $\alpha=f(\beta)$,

$$(r^2-a^2)/2x=f\{(r^2-c^2)/2y\}.$$

The algebraical form of $f(u)$ which can give a cubic surface is $A+Bu$, and the equation of the surface is of the form

$$C\,(r^2-a^2)/x+D\,(r^2-c^2)/y=1.$$

(2) The functional equation of a right conoid, whose axis is Oz, is $F(z,y/x)=0$, and for the right conoid of the n^{th} degree, the equation is $Z_{n-r}x^r+Z'_{n-r}x^{r-1}y+\ldots=0$, where Z_s is any integral function of z of the s^{th} degree.

For a given value of z, y/x has r values, and the least value of $n-r$ is 1, otherwise z would disappear.

(3) The equations of a generating line may be written

$$n\,(y-b)-m\,(z-c)=\alpha\,\{n\,(x-a)-l\,(z-c)\},\ \text{ and }\ z=\beta x+\gamma y,$$

and in order that this line may generate a ruled surface, β and γ must be functions of α, whence the equation given in the problem.

(4) The directions (λ,μ,ν) of the tangent lines to the two branches of the curves of intersection with the tangent plane are given by $\lambda U=\ldots=0$ and $\lambda^2u+\ldots+2\mu\nu u'+\ldots=0$, and the required equation is the condition that the two directions should be at right angles, Art. 26.

(5) Taking the axis of z for the axis, and the equation $z=mx$ for that of the director plane of any conoid, the equation of the family of such conoids is $z=mx+f(y/x)\equiv mx+f(u)$.

The condition corresponding to that of (4) is

$$(1+q^2)\,r-2pqs+(1+p^2)\,t=0,$$

whence shew that $2uf'(u)+2m\,\{f'(u)\}^2/x+(u^2+1+m^2)f''(u)=0$, thus $f(u)$ is a function of u, only when $m=0$, or when the conoid is a right conoid. In this case

$$f''(u)/f'(u)=-2u/(u^2+1),\quad f'(u)=c/(u^2+1);$$

$$\therefore f(u)=c\tan^{-1}u=z, \text{ or } y=x\tan(z/c).$$

(6) The equations of a generating line are

$$y=\alpha x,\quad z=\beta\,(c-r),\quad \text{where } r^2=x^2+y^2,$$

and $\beta=f(\alpha)$ gives the functional equation $z=(c-r)f(y/x)$.

Find p and q, and shew that $px+qy=-rf(y/x)$. (1)

The osculating plane of the geodesic at $(x', y', 0)$ contains the normal, and its trace on the plane xy touches the circle, these conditions are represented by

$$A\,(x-x'+p'z)+B\,(y-y'+q'z)=0,\quad \text{and}\quad A/x'=B/y';$$

$$\therefore\ x'x+y'y-c^2=-(p'x'+q'y')\,z=czf(y'/x'), \text{ by (1)}.$$

(7) The torse is the envelope of a plane which touches both curves, and therefore contains the tangents to both, these tangents must therefore be parallel, and their equations must be

$$y=mx+a/m,\ z=0;\ \text{and}\ x=y/m+ma,\ z=c;$$

hence, the equation of the enveloping plane is

$$y-mx-a/m+Az\equiv y-mx+m^2a+A\,(z-c)=0,$$

so that $Ac-m^2a=a/m$, and the equation of the plane is

$$my-m^2x-a+(m^3+1)\,az/c=0,\quad (1)$$

and from those of the next two consecutive planes

$$y-2mx+3m^2az/c=0,\quad (2)$$

$$\text{and}\ -2x+6maz/c=0\ \text{or}\ m=\tfrac{1}{3}cx/az.$$

The equations of the edge are obtained by substituting for m in (1) and (2).

See H. M. Taylor 'On the generation of a torse through two given curves,' *Mess. of Math.* vol. V. p. 1, where he gives this problem as an illustration.

(8) Let $x^2/a+y^2/b+z^2/c=1$ be the equation of the ellipsoid, and let (ξ, η, ζ) be the point Q in the normal at $P\ (x, y, z)$, $PQ=\lambda/p$, then $\xi-x=\lambda x/a$, &c. Hence the equation of the locus of Q is $a\xi^2/(a+\lambda)^2+\ldots=1$. (1)

For the envelope of the ellipsoid (1) eliminate λ from (1) and $a\xi^2/(a+\lambda)^3+\ldots=0$, (2). By Art. 721, the coordinates ξ, η, ζ of

the two centres of curvature at P are the two sets of values of $x(a+k)/a$, &c. where $x^2/(a+k)+\ldots=1$, or $x^2/a(a+k)+\ldots=0$, hence the equation of the surface of centres is found by the elimination of k from the equations

$$a\xi^2/(a+k)^2+\ldots=1 \text{ and } a\xi^2/(a+k)^3+\ldots=0,$$

which proves the second theorem.

LXIV.

(1) The functional equation of such surfaces is

$$b^2\{x-f(z)\}^2+a^2\{y-\phi(z)\}^2=a^2b^2; \quad (1)$$

$$\therefore\ b^2\{x-f(z)\}+Wp=0, \quad a^2\{y-\phi(z)\}+Wq=0,$$

$$\text{where } W\equiv -b^2f'(z)\{x-f(z)\}-a^2\phi'(z)\{y-\phi(z)\},$$

$$\text{hence } qb^2\{x-f(z)\}-pa^2\{y-\phi(z)\}=0; \quad (2)$$

$$\therefore\ sb^2\{x-f(z)\}-ra^2\{y-\phi(z)\}+qb^2+wp=0,$$

$$\text{and } tb^2\{x-f(z)\}-sa^2\{y-\phi(z)\}-pa^2+wq=0,$$

$$\text{where } w\equiv -qb^2f'(z)+pa^2\phi'(z);$$

$$\therefore\ (qs-pt)\,b^2\{x-f(z)\}-(qr-ps)\,a^2\{y-\phi(z)\}+q^2b^2+p^2a^2=0,$$

$$\text{by (2) and (1), } x-f(z)=a^2p\,(a^2p^2+b^2q^2)^{-\frac{1}{2}},$$

$$\text{and } y-\phi(z)=b^2q\,(a^2p^2+b^2q^2)^{-\frac{1}{2}}; \quad (3)$$

$$\therefore\ a^2b^2(q^2r-2pqs+p^2t)-(a^2p^2+b^2q^2)^{\frac{3}{2}}=0. \quad (4)$$

The two first integrals of (4) may be obtained by eliminating separately $f(z)$ and $\phi(z)$ from (1) and (2), and so obtaining (3).

(2) Let the equations of the internal and external surfaces be

$$x^2/a^2+y^2/b^2+z^2/c^2=1\pm\lambda,$$

the shell being thin, the thickness at any point is λp, which is the difference between the perpendiculars upon parallel tangent planes, hence p is constant at all points for which the thickness is constant. The problem is to find the envelope of the plane $lx+my+nz=0$, l, m, n being subject to the condition

$$l^2a^2+m^2b^2+n^2c^2=p^2(l^2+m^2+n^2),$$

$$\text{hence } l(a^2-p^2)/x=m(b^2-p^2)/y=n(c^2-p^2)/z;$$

$$\therefore\ x^2/(a^2-p^2)+y^2/(b^2-p^2)+z^2/(c^2-p^2)=0.$$

(3) Let $b\eta^2+c\zeta^2=2\xi$ be the equation of the paraboloid, (l, m, n) the direction of a chord whose middle point is (x, y, z) and length $2r$; then $bmy+cnz=l$ (1), and $(bm^2+cn^2)\,r^2=2x-by^2-cz^2$, hence the equation of the sphere whose envelope is required is

$$(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2=r^2=k(2x-by^2-cz^2), \quad (2)$$

where $k^{-1}=bm^2+cn^2$; therefore, by Art. 874, y and z being connected by (1),

$$\xi-x+k=0, \ (3) \quad \text{and} \quad (\eta-y-kby)/bm=(\zeta-z-kcz)/cn, \ (4)$$

we have to eliminate x, y, z from (1), (2), (3), (4);

writing y' for $y-\eta/(1+kb)$ and z' for $z-\zeta/(1+kc)$,

(4) becomes $(1+kb)\,y'/bm=(1+kc)\,z'/cn\equiv\rho$, suppose,

(1) becomes $bmy'+cnz'=l-bm\eta/(1+kb)-cn\zeta/(1+kc)\equiv u$,

(2) becomes

$$(1+kb)\,y'^2+(1+kc)\,z'^2=2k\xi+k^2-kb\eta^2/(1+kb)-kc\zeta^2/(1+kc)\equiv v,$$

$$\therefore\ \rho\,\{b^2m^2/(1+kb)+c^2n^2/(1+kc)\}=u,$$

$$\rho^2\,\{b^2m^2/(1+kb)+c^2n^2/(1+kc)\}=v;$$

$$\therefore\ u^2=\{b^2m^2/(1+kb)+c^2n^2/(1+kc)\}\,v.$$

Hence the envelope is a paraboloid or a parabolic cylinder; and it may be shewn, since c is negative, that the paraboloid will be elliptic or hyperbolic, as $-(bm^2/c+cn^2/b)<$ or >1.

If the original paraboloid had been elliptic the envelope would have been an elliptic paraboloid.

(4) Since $\frac{1}{2}U=ax+c'y+b'z+a''$, $\frac{1}{2}u=a$, $\frac{1}{2}u'=a'$, &c., the condition of perpendicularity of the generating lines is

$$a\,(V^2+W^2)+\ldots-2a'VW-\ldots=0. \quad (1)$$

By transformation of coordinates let the equation of the surface become $\alpha x^2+\beta y^2+\gamma z^2+2\alpha''x+\delta=0$, the condition (1) becomes

$$(\beta+\gamma)\,(\alpha x+\alpha'')^2+(\gamma+\alpha)\,\beta^2y^2+(\alpha+\beta)\,\gamma^2z^2=0,$$

the coefficient of $x^2=\alpha\,(I_2-\beta\gamma)=\alpha I_2-\Delta$, see Art. 413, and the terms of the second degree are $I_2\,(\alpha x^2+\ldots)-\Delta\,(x^2+\ldots)$, which with the original coordinates gives

$$I_2\,(ax^2+\ldots+2a'yz+\ldots)-\Delta\,(x^2+y^2+z^2);$$

also in (1) the coefficient of x is

$$2a''\,(ab+ac-b'^2-c'^2)+2b''\,(cc'-a'b')+2c''\,(bb'-a'c')$$

$$=2a''I_2-2\,(Aa''+C'b''+B'c'')=2a''I_2+dH/da'', \text{ see Arts. 391, 392.}$$

Hence, at the intersection of the given surface with the surface (1),

$$\Delta\,(x^2+y^2+z^2)-x\,dH/da''-y\,dH/db''-z\,dH/dc''$$
$$+I_2d-I_1\,(a''^2+b''^2+c''^2)+aa''^2+\ldots+2a'b''c''+\ldots=0,$$

shewing that the points lie on a sphere.

(5) Take the axis of revolution for that of z, and let the equation of the surface be $x^2+y^2=\phi^2$, where ϕ is any function of z only. The condition given is represented by

$$(1+q^2)\,r-2pqs+(1+p^2)\,t=0.$$

$$x=p\phi\phi',\ y=q\phi\phi',\ \therefore\ 1=(p^2+q^2)\,\phi'^2, \quad (1)$$

$$1=r\phi\phi'+p^2\,(\phi\phi''+\phi'^2),$$

$$0=s\phi\phi'+pq\,(\phi\phi''+\phi'^2),$$

$$1=t\phi\phi'+q^2\,(\phi\phi''+\phi'^2);$$

$$\therefore\ 2+p^2+q^2=(p^2+q^2)(\phi\phi''+\phi'^2);$$

$\therefore$, by (1), $2\phi'^2+1=\phi\phi''+\phi'^2$, and $2\phi'\phi''\phi^{-2}-2\phi'^3\phi^{-3}=2\phi'\phi^{-3}$,

hence $\phi'^2\phi^{-2}=c^{-2}-\phi^{-2}$, and $dz/d\phi=c(\phi^2-c^2)^{-\frac{1}{2}}$;

$$\therefore\ \phi=\tfrac{1}{2}c\,(e^{z/c}+e^{-z/c}).$$

(6) Let the origin be the vertex of the cone, (f, g, h) the centre of the conicoid, $a(x-f)^2+\ldots=1$ its equation, and

$$\alpha x+\beta y+\gamma z=1,\ \alpha' x+\beta' y+\gamma' z=1$$

the equations of the planes of the sections. The equation of the cone is

$$a(x-f)^2+b(y-g)^2+c(z-h)^2-1-\rho(\alpha x+\beta y+\gamma z-1)(\alpha' x+\beta' y+\gamma' z-1)=0.$$

If the axis of z be an axis of the cone the only terms of the equation are those involving x^2, y^2, z^2, and xy;

$$\therefore\ af^2+bg^2+ch^2-1=\rho,\ 2af=\rho(\alpha+\alpha'),$$

$$2bg=\rho(\beta+\beta'),\ 2ch=\rho(\gamma+\gamma'),\ \alpha\gamma'+\alpha'\gamma=0,\ \beta\gamma'+\beta'\gamma=0,$$

$$\therefore\ \alpha'/\alpha=-\gamma'/\gamma=\beta'/\beta=\sigma, \text{ suppose};$$

$$\therefore\ (\sigma+1)\alpha=2af/\rho=A,\ (\sigma+1)\beta=B,\ -(\sigma-1)\gamma=C,$$

where A, B, C are constants; hence the equation of one of the planes is $(\sigma-1)(Ax+By)-(\sigma+1)Cz=\sigma^2-1$, and the envelope of such planes has the equation

$$(Ax+By-Cz)^2=4(Ax+By+Cz-1).$$

Turning the axes of x and y through an angle $\tan^{-1}(B/A)$, the equation is $\{x'\sqrt{(A^2+B^2)}-Cz\}^2=4\{x'\sqrt{(A^2+B^2)}+Cz-1\}$;

$$\text{let } \sqrt{(A^2+B^2)}=D\sin\phi,\ C=D\cos\phi;$$

$$\therefore\ D(x'\sin\phi-z\cos\phi)^2=4(x'\sin\phi+z\cos\phi-D^{-1}).$$

In fig. 9 let ORM and RN be the lines whose equations are

$$x'\sin\phi-z\cos\phi=0 \text{ and } x'\sin\phi+z\cos\phi-D^{-1}=0.$$

Draw PM, PN parallel to RN and RM, then

$$PM\sin 2\phi=-x'\sin\phi+z\cos\phi,\ PN\sin 2\phi=x'\sin\phi+z\cos\phi-D^{-1};$$

$$\therefore\ D\sin 2\phi\, PM^2=4PN,$$

and if S be the focus, $SR^{-1}=D\sin 2\phi$; the coordinates of R are $\frac{1}{2}D^{-1}\operatorname{cosec}\phi$ and $\frac{1}{2}D^{-1}\sec\phi$, $\therefore$ OR, which is perpendicular to

$$x'\cos\phi+z\sin\phi=0, \text{ is } \tfrac{1}{2}D^{-1}(\cot\phi+\tan\phi)=D^{-1}/\sin 2\phi;$$

$\therefore$ $SR=OR$, hence the directrix passes through O. The envelope is therefore a parabolic cylinder whose directrix passes through the fixed point.

(7) The torse is the envelope of the plane $x\xi+y\eta+z\zeta=r^2$, (1) subject to the conditions

$$x^2+y^2+z^2=r^2,\ (2)\ \text{ and }\ ax^2+by^2+cz^2=0,\ (3)$$

which determine the sphero-conic. (1) is the equation of the tangent plane at (x, y, z), and if $(x+dx, y+dy, z+dz)$ be a

consecutive point on the sphero-conic,

$$x\,dx/(b-c)=y\,dy/(c-a)=z\,dz/(a-b);$$

at the intersection of the tangent planes at these consecutive points

$$(b-c)\,\xi/x+(c-a)\,\eta/y+(a-b)\,\zeta/z=0; \quad (4)$$

(1) and (4) are the equations of the generating line of the torse through (x, y, z), and for the consecutive generating line

$$(b-c)^2\,\xi/x^3+(c-a)^2\,\eta/y^3+(a-b)^2\,\zeta/z^3=0; \quad (5)$$

hence, by (4) and (5), at the edge of regression

$$(b-c)\,\xi/ax^3=(c-a)\,\eta/by^3=(a-b)\,\zeta/cz^3=\rho,$$

and, eliminating ζ from (1) and (4),

$$\{(a-b)\,x^2-(b-c)\,z^2\}\,\xi/x+\{(a-b)\,y^2-(c-a)\,z^2\}\,\eta/y=(a-b)\,r^2,$$

$$\therefore, \text{ by (3)}, \; b-a=b\,\xi/x-a\,\eta/y=ab\rho\,\{x^2/(b-c)-y^2/(c-a)\}$$
$$=abcr^2\rho/(b-c)\,(c-a);$$

$$\therefore\; x^3=cr^2b\xi/(b-a)\,(c-a), \quad y^3=cr^2\,a\eta/(b-a)\,(b-c),$$

$$\text{and } (c-a)^{\frac{1}{3}}\,(b\xi)^{\frac{2}{3}}+(c-b)^{\frac{1}{3}}\,(a\eta)^{\frac{2}{3}}=(b-a)^{\frac{2}{3}}\,(cr^2)^{\frac{1}{3}}.$$

(8) The equation of a sphere of the system is

$$x^2+y^2+z^2-2\alpha x+\alpha^2y/2a=0,$$

and that of the envelope is $y\,(x^2+y^2+z^2)=2ax^2$;

$$\text{hence } pz=2ax/y-x, \quad qz=-ax^2/y^2-y,$$

$$rz=2a/y-1-p^2, \; sz=-2ax/y^2-pq, \; tz=2ax^2/y^3-1-q^2.$$

The differential equation of the projections of the lines of curvature are given in Art. 718; now in this case

$$\{(1+q^2)\,s-pqt\}\,z=-(1+q^2)\,(2ax/y^2+pq)-pq\,(2ax^2/y^3-1-q^2)$$
$$=-\,2ax\,y^{-3}\,\{y+q\,(px+qy)\},$$

$$\{(1+q^2)\,r-(1+p^2)\,t\}\,z=(1+q^2)(2a/y-1-p^2)-(1+p^2)(2ax^2/y^3-1-q^2)$$
$$=2a\,y^{-3}\,(y^2-x^2+q^2y^2-p^2x^2),$$

$$\{pqr-(1+p^2)\,s\}\,z=pq\,(2a/y-1-p^2)+(1+p^2)\,(2ax/y^2+pq)$$
$$=2a\,y^{-2}\,\{x+p\,(px+qy)\};$$

hence the differential equation becomes

$$(y\,dx-x\,dy)\,[\{y+q\,(px+qy)\}\,dy+\{x+p\,(px+qy)\}\,dx]=0,$$

therefore the differential equations of the two systems become

$$y\,dx-x\,dy=0 \text{ and } (2y^3+yx^2-ax^2)\,x\,dy-(2y^3+yx^2-2ax^2)\,y\,dx=0.$$

The integral of the first is $y=Cx$, the corresponding lines of curvature being plane curves, and for that of the second let $y=vx$, whence $(2v^2+1)\,x^2v\,dv+a\,(v\,dx-x\,dv)=0$,

$$\therefore\; v^4+v^2-2av/x=D, \text{ and } y^4+x^2\,(y^2-2ay)=Dx^4.$$

(9) The cubic must be supposed not to be made up of surfaces of a lower degree, as of a plane and cone, which may in one sense be called a torse.

We have to shew that the edge of regression cannot be of double curvature, for in that case four generators P, Q, R, and S may be found which do not intersect, a straight line T can then be found which will intersect all four, and therefore will lie entirely on the torse; the tangent plane at any point of P must therefore contain T as well as the next consecutive generator P', so that P' will lie in the plane (P, T); similarly for the succeeding consecutive generator, until Q is shewn to lie in the same plane, which is contrary to the supposition that P and Q do not intersect.

LXV.

(1) If $u=0$, $v=0$ be the equations of two of the surfaces satisfying the conditions, that of any one of the cluster will be $\lambda u+\mu v=0$, and for its r^{th} polar with respect to (x', y', z', w'), $\lambda D^r u+\mu D^r v=0$, and all the r^{th} polars will have a common curve, $D^r u=0$, $D^r v=0$.

(2) If $u=0$, $v=0$, $w=0$ be the equations of three surfaces through the points, $\lambda u+\mu v+\nu w=0$ will be that of any other surface of the cluster; hence all the r^{th} polars have as common points the intersections of the three surfaces $D^r u=0$, $D^r v=0$, and $D^r w=0$, which are $(n-r)^3$ in number.

(3) Let (x', y', z', w') and (x'', y'', z'', w'') be P and Q; and let $u=0$ be the equation of the surface, those of U and V are

$$\left(x'\frac{d}{dx}+\ldots\right)u=0, \text{ and } \left(x''\frac{d}{dx}+\ldots\right)u=0;$$

the analytical statement of the theorem is

$$(x'd/dx+\ldots)(x''d/dx+\ldots)u\equiv(x''d/dx+\ldots)(x'd/dx+\ldots)u.$$

(4) Proceeding as in (3), shew that

$$(x'd/dx+\ldots)^p(x''d/dx+\ldots)^q u\equiv(x''d/dx+\ldots)^q(x'd/dx+\ldots)^p u.$$

(5) Let $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ be the points P and Q, and let $u=0$ be the equation of the surface of the n^{th} degree.

The p^{th} polar of P is $(d/dx)^p u=0$, and if this surface have a double point at Q, its equation will have none of the four terms involving y^{n-p}, xy^{n-p-1}, zy^{n-p-1}, or wy^{n-p-1}, hence the equation $u=0$ will be without those involving

$$x^p y^{n-p},\ x^{p+1}y^{n-p-1},\ x^p zy^{n-p-1}, \text{ or } x^p wy^{n-p-1}.$$

The equation of the $(n-p-1)^{\text{th}}$ polar of Q is $(d/dy)^{n-p-1}u=0$, which will have no terms involving $x^p y$, x^{p+1}, $x^p z$, or $x^p w$, the polar of Q will therefore have a double point at P.

(6) The equation of the cubic surface on which CD of the fundamental tetrahedron is a double line may be written

$$F\equiv x^3+x^2(ay+bz+cw)+xy(a'y+b'z+c'w)+y^2(b''y+c''z+d''w)$$
$$\equiv x^3+x^2u+xyu'+y^2u''=0, \quad (1)$$

and a line through $A(1, 0, 0, 0)$ meets the surface in three coincident points given by the three equations $F=0$, $DF=0$, and $D^2F=0$, where $D \equiv d/dx$, Art. 909,

$$\therefore\ 3x^2+2xu+yu'=0, \quad 6x+2u=0;$$

eliminating x from these equations and (1) we obtain

$$y(9yu''-uu')=0, \text{ and } u^2=3yu';$$

these equations give the six generators of the enveloping cone, whose vertex is A, which touch at three coincident points, forming, in the case of the general cubic surface, cuspidal edges; $y=0$, and $u^2=3yu'$ give two coincident points on CD, the remaining four points are the points of intersection of the two conics $9yu''=uu'$ and $u^2=3yu'$, one point is on CD, and the other three correspond to the proper cuspidal edges. The three points on CD are at the intersection of CD by the third line which, with the double line, makes up the section of the cubic surface by the plane ACD.

(7) Working with the corresponding surface $x^{-1}+y^{-1}+\ldots=0$, or $yzw+zxw+xyw+xyz=0$, the polar conicoid with respect to (x', y', z', w') is given by the equation

$$(y'+z')xw+(z'+x')yw+(x'+y')zw+(x'+w')yz$$
$$+(y'+w')zx+(z'+w')xy=0; \quad (1)$$

if this represent two planes, it must be

i. one of the three forms, such as $(Ax+By)(Cz+Dw)=0$,

or ii. one of the four forms, such as $x(By+Cz+Dw)=0$.

i. If the terms in xy and zw be wanting, $z'+w'=0$ and $x'+y'=0$, and $A/B=(y'+w')/(x'+w')=(y'+z')/(z'+x')$, $\therefore (z'+x')^2=(z'-x')^2$, hence $x'=0$ or $z'=0$; if $x'=0$, $y'=0$ and $z'+w'=0$; if $z'=0$, $w'=0$ and $x'+y'=0$.

ii. If the terms in yw, zw, and yz be wanting,

$$z'+x'=0, \quad x'+y'=0, \text{ and } x'+w'=0; \ \therefore -x'=y'=z'=w'. \quad (2)$$

The polar plane is given by interchanging dashed and undashed letters in (1), and its equation, by (2), becomes $3x-y-z-w=0$.

Writing x/l and x'/l for x and x', &c., we have the four positions of P and the polar plane for the given surface.

i. supplies six more positions in the six edges, and the corresponding polar planes $x/l+y/m=0$, &c.

(8) This follows immediately from Art. 903.

LXVI.

(1) The equation of the tangent plane at (x, y, z) to the surface $u \equiv u_n+u_{n-1}+\ldots=0$ is $U\xi+V\eta+W\zeta=-u_{n-1}-2u_{n-2}-\ldots$, and the perpendicular from any point (x', y', z') upon it is constant,

$$\therefore (Ux'+\ldots+u_{n-1}+2u_{n-2}+\ldots)^2=C^2(U^2+V^2+W^2).$$

(2) Let (α, β, γ) be a point on the surface, we have to find the envelope of the first polar whose equation is

$$\alpha\, dU/dx + \beta\, dU/dy + \gamma\, dU/dz = P,$$

subject to the condition $\alpha^m/a^m + \beta^m/b^m + \gamma^m/c^m = 1$,

$$\therefore\ dU/dx = P\alpha^{m-1}/a^m, \text{ \&c.},$$

$$\therefore\ (a\, dU/dx)^{\frac{m}{m-1}} + \ldots = P^{\frac{m}{m-1}}(\alpha^m/a^m + \ldots) = P^{\frac{m}{m-1}}.$$

(3) The equation of the first polar is $\Sigma\{xw(ny' + mz')\} = 0$; if this represent a sphere it must be the sphere circumscribing the fundamental tetrahedron, hence, by Art. 587,

$$a'^2 = \sigma(ny' + mz'), \qquad a^2 = \sigma(rx' + lw'),$$
$$b'^2 = \sigma(lz' + nx'), \qquad b^2 = \sigma(ry' + mw'),$$
$$c'^2 = \sigma(mx' + ly'), \qquad c^2 = \sigma(rz' + nw');$$

hence $mna^2 + lra'^2 = \sigma lmnr(x'/l + y'/m + z'/n + w'/r)$, (1)

$y'z'a^2 + x'w'a'^2 = \sigma x'y'z'w'(l/x' + m/y' + n/z' + r/w')$, (2)

and $b'^2/ln + c'^2/lm - a'^2/mn = 2\sigma x'/l$; (3)

by (1), $mna^2 + lra'^2 = lnb^2 + mrb'^2 = lmc^2 + nrc'^2 = \rho$,

by (2), $yza^2 + xwa'^2 = \ldots = \ldots$ is the locus of the poles,

by (3), since $la'^2 = \rho/r - a^2mn/r$,

$$\therefore\ 2\sigma x' = b'^2/n + c'^2/m + a^2/r - \rho/mnr,$$

whence the ratios $x' : y' : z' : w'$.

(4) The degree of the enveloping cone is $n(n-1)$, the number of sides of the cone which meet the surface in three consecutive points $= n(n-1)(n-2)$, Art. 909, the number which touch the surface at two points is $\frac{1}{2}n(n-1)(n-2)(n-3)$, Art. 915. Hence the number of cuspidal edges of the enveloping cone, and therefore of cusps on the plane section of the cone, is $n(n-1)(n-2) = \sigma$; the number of double sides of the cone, and therefore of multiple points on the plane section, is $\frac{1}{2}n(n-1)(n-2)(n-3) = \lambda$; the degree of the plane section is $n(n-1)$, hence, by Art. 670, the class is $n(n-1)\{n(n-1)-1\} - 2\lambda - 3\sigma = n(n-1)^2$, which is also shewn in Art. 917. Therefore the number of points of inflexion of the plane section

$$= \sigma + 3\{n(n-1)^2 - n(n-1)\} = n(n-1)\{n-2+3(n-2)\} = 4n(n-1)(n-2).$$

(5) If CD, an edge of the fundamental tetrahedron, lie entirely on the surface, the equation of the surface will be of the form $F \equiv x\phi + y\psi = 0$, where ϕ and ψ are functions of the $(n-1)^{\text{th}}$ degree, which become ϕ_0 and ψ_0 when $x = 0$ and $y = 0$.

Let (x', y', z', w') be a parabolic point P, then, Art. 906, the polar conicoid $D'^2(x'\phi' + y'\psi') = 0$ is a cone.

If u_{11}, u_{12}, &c. be written for d^2F'/dx'^2, $d^2F'/dx'dy'$, &c. the parabolic points lie in the surface which is the Hessian of

$$u_{11}x^2+\ldots+2u_{23}yz+\ldots+2u_{14}xw+\ldots=0,$$

we have to find the points in which this surface meets CD.

When $x'=0$ and $y'=0$, $u_{13}=d\phi'_0/dz'$, $u_{14}=d\phi'_0/dw'$, $u_{23}=d\psi'_0/dz'$, $u_{24}=d\psi'_0/dw'$, and $u_{33}=0=u_{34}=u_{44}$; hence, at the point of intersection with CD,

$$\begin{vmatrix} u_{11}, & u_{12}, & u_{13}, & u_{14} \\ u_{12}, & u_{22}, & u_{23}, & u_{24} \\ u_{13}, & u_{23}, & 0, & 0 \\ u_{14}, & u_{24}, & 0, & 0 \end{vmatrix} \equiv (u_{13}u_{24}-u_{23}u_{14})^2=0.$$

Therefore CD touches the parabolic curve in $2(n-2)$ points.

(6) Let $(\xi, \eta, \zeta, \omega)$ be the pole of the tangent plane at the point (x', y', z', w'), whose equation must therefore be

$$\xi x+\eta y+\zeta z+\omega w=0, \quad (1)$$

$$\therefore\ 2az'x'+2bw'y'=\rho\xi, \quad 2bx'w'+2cz'y'=\rho\eta, \quad (2)$$

$$ax'^2+cy'^2=-2bw'x'y'/z'=\rho\zeta, \quad \text{and} \quad 2bx'y'=\rho\omega. \quad (3)$$

By (3), $z'\zeta+w'\omega=0$, and by (1), $x'\xi+y'\eta=0$,

by (2), $\eta(az'x'+bw'y')-\xi(bx'w'+cz'y')=0$,

$$\therefore\ \eta(a\omega\eta+b\zeta\xi)+\xi(b\eta\zeta+c\omega\xi)=0,$$

$$\text{or}\ \omega(a\eta^2+c\xi^2)+2b\xi\eta\zeta=0.$$

CD is a double line on the given surface, and, by Art. 924, the class is lowered by $7.3-12\equiv 9$, hence the degree of the reciprocal is $3.2^2-9\equiv 3$.

(7) and (8) A solution of these problems is given in Salmon's geometry of three dimensions, Arts. 588 and 598, in connection with which his Arts. 473 and 474 should be studied. (7) was first given in *Camb. and Dublin Math. Jour.*, vol. IV. p. 258, and (7) and (8) afterwards in *Quart. Jour.* vol. I. pp. 333 and 337.

LXVII.

(1) The volume is $\iiint dx\,dy\,dz$ or $\iiint \rho\,d\rho\,d\theta\,dz$ taken from $z=z_1$ to z_2, $\rho=0$ to $2r\cos\theta$, $\theta=-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, and $z_2-z_1=(a\sim a')x/c$; the volume is

$$(a\sim a')c^{-1}\iint\rho^2\cos\theta\,d\rho\,d\theta=\tfrac{2}{3}(a\sim a')c^{-1}\int_0^{\frac{1}{2}\pi}8r^3\cos^4\theta\,d\theta=\pi r^3(a\sim a')/c.$$

(2) Let A be the area of a section of the surface by the plane $x+y+z=p\sqrt{3}$; this section, from symmetry, is a circle, the distance of whose centre from the origin is p; and when $x=y=z$, $p^2=x^2+y^2+z^2=a^2$; hence the volume of the surface is $\int_a^c A\,dp$.

The equation of the projection of the section on the plane xy is

$$xy+(x+y)(p\sqrt{3}-x-y)=a^2,$$

turning the axes through $\frac{1}{4}\pi$, the equation becomes

$$\tfrac{1}{2}(x^2 - y^2) - 2x^2 + xp\sqrt{6} = a^2,$$

$$\text{or } 3x^2 - 2px\sqrt{6} + 2p^2 + y^2 = 2(p^2 - a^2),$$

$$\therefore A\sqrt{\tfrac{1}{3}} = 2\pi\, 3^{-\frac{1}{2}}(p^2 - a^2),$$

$$\text{hence the volume} = 2\pi\int_a^c (p^2 - a^2)\,dp = \tfrac{2}{3}\pi(c-a)^2(c+2a).$$

(3) The integrations are to be taken from $x=(y^2+z^2)/4a$ to $x=z+a$, from $y=-\sqrt{\{8a^2-(z-2a)^2\}}$ to $y=+\sqrt{\{8a^2-(z-2a)^2\}}$, and from $z-2a=-a\sqrt{8}$ to $+a\sqrt{8}$, giving for the volume

$$\int \tfrac{1}{3}a^{-1}dz\,\{8a^2 - (z-2a)^2\}^{\frac{3}{2}},$$

which becomes, if $z - 2a = a\sqrt{8}\sin\theta$, $\frac{1}{3}2^7a^3\int_0^{\frac{1}{2}\pi}\cos^4\theta\,d\theta = 8\pi a^3$.

(4) The volume, including the part below as well as that above the plane of xy, is $2\iiint r\,dr\,d\theta\,dz$, the integrations being taken from $z=0$ to $mr\cos\theta$, from $r=0$ to a, from $\theta=0$ to $\frac{1}{2}\pi$.

(5) Representing the bounding surfaces by cylindrical coordinates, $r^2=az$, $r=a\cos\theta$, and $z=0$, the volume is $\iiint r\,dr\,d\theta\,dz$, the integrations from $z=0$ to r^2/a, $r=0$ to $a\cos\theta$, $\theta=-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$.

(6) The volume $=\iiint r\,dr\,d\theta\,dz$, the integrations being taken from

$$z=0 \text{ to } \tfrac{1}{2}r^2(a\cos^2\theta + b\sin^2\theta),\ r=0 \text{ to } 2c\cos\theta,\ \theta=-\tfrac{1}{2}\pi \text{ to } \tfrac{1}{2}\pi,$$

$$=\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\tfrac{1}{8}(2c\cos\theta)^4(a\cos^2\theta + b\sin^2\theta)\,d\theta = \tfrac{1}{8}\pi c^4(5a+b).$$

(7) Let $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ be the equation of the ellipsoid, $x=h$ that of the plane of the base of one of the cones; the area of the base $=C(a^2-h^2)$ where C is constant for all values of h, the volume of the cone is $\frac{1}{3}Ch(a^2-h^2)$, and that of the segment of the ellipsoid is

$$\int_h^a C(a^2-\alpha^2)\,d\alpha = C\{a^2(a-h) - \tfrac{1}{3}(a^3-h^3)\} = \tfrac{1}{3}C(a-h)(2a^2-ah-h^2),$$

$\therefore$ the volume contained within each sheet of the cone $=\frac{2}{3}Ca^2(a-h)$, hence the volume between the two cones (h) and $(h') \propto h \sim h'$.

(8) Let p be the perpendicular on the tangent plane at the point (x, y, z) at which ΔS is situated, then $\Delta S = dx\,dy\,.c^2/pz$, also $Ap = \pi abc$; let $x = ax'$, &c.,

$$\therefore \Delta S/A = c\,dx\,dy/\pi abz = dx'dy'/\pi z' = r\,dr\,d\theta/\pi\sqrt{(1-r^2)},$$

$$\therefore \Sigma(\Delta S/A) = 4\int_0^1 r\,dr\,(1-r^2)^{-\frac{1}{2}} = 4.$$

If $x = a\cos\alpha$, $r=\sin\alpha$, $y = br\cos\beta$, $z = cr\sin\beta$,

$$dy\,dz = bcr\,dr\,d\beta = bc\sin\alpha\cos\alpha\,d\alpha\,d\beta,$$

and

$$\Delta S = dy\,dz.a^2/px = abc\sin\alpha\,d\alpha\,d\beta\sqrt{\{\sin^2\alpha(\cos^2\beta/b^2 + \sin^2\beta/c^2) + \cos^2\alpha/a^2\}}.$$

LXVIII.

(1) The volume is composed of four equal portions, viz. those for which xyz is positive.

Let $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$,

$\therefore\ r=c\sin^2\theta\cos\theta\sin\phi\cos\phi$ is the equation of the surface.

The volume

$$=4\iiint r^2\sin\theta\,d\phi\,d\theta\,dr=\tfrac{4}{3}c^3\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\sin^7\theta\cos^3\theta\sin^3\phi\cos^3\phi\,d\theta\,d\phi.$$

(2) When y is constant the section is an ellipse

$$(ax-y^2)^2+(bz-y^2)^2=2y^4,$$

whose area is $2\pi y^4/ab$, hence the volume is $\int_{-k}^{k}2\pi y^4dy/ab$.

(3) Let the plane of the disc be parallel to zOx, C its centre on a circle in the plane xOy, ACB the diameter parallel to Ox cutting Oy in M, MP the ordinate in the plane yOz; and let $\angle COx=\theta$, $\therefore\ CM=c\cos\theta$ and $PM=OM=c\sin\theta$. The portion of the cavity in the compartment $Oxyz$ is generated by the part of the semicircle MPB, hence the entire volume is

$$8\int_0^c d(c\sin\theta)\left\{\tfrac{1}{2}c^2(\pi-\theta)+\tfrac{1}{2}c^2\sin\theta\cos\theta\right\}$$

$$=4c^3\left(\pi\sin\theta-\theta\sin\theta-\cos\theta-\tfrac{1}{3}\cos^3\theta\right)_0^{\frac{1}{2}\pi}=\tfrac{2}{3}c^3(3\pi+8).$$

(4) The volume $\iiint dx\,dy\,dz$ will be obtained by summing in the order x, y, z; the first summation gives the parallelepiped $dx\,dy(z_2-z_1)$, z_1, z_2 being the two values of z for given values of x and y; the second gives an elliptic disc, as the sum of the parallelepipeds for a given value of x taken from $y=y_1$ to y_2, y_1, y_2 being the values of y for which the parallelepiped vanishes, determined by the equation $z_2-z_1=0$; the third summation gives the volume, being taken from the value $-x_1$ to $+x_1$ for which the area of the elliptic disc vanishes.

The calculation of the values of these limits is as follows:

$$cz^2+2(a'y+b'x)z+ax^2+by^2+2c'xy-1\equiv c(z-z_1)(z-z_2),$$

$$\text{whence }c^2(z_2-z_1)^2=4\{(a'y+b'x)^2-c(ax^2+by^2+2c'xy-1)\}$$

$$\equiv4(bc-a'^2)(y-y_1)(y_2-y)\equiv4A(y-y_1)(y_2-y),\ \text{Art. 321},$$

$$\text{whence }A^2(y_2-y_1)^2=4c(A-\Delta x^2)\equiv4c\Delta(x_1^2-x^2).$$

Hence the volume $=\int_{-x_1}^{x_1}dx\int_{y_1}^{y_2}2c^{-1}A^{\frac{1}{2}}\sqrt{\{\tfrac{1}{4}(y_2-y_1)^2-(y-\tfrac{1}{2}y_1-\tfrac{1}{2}y_2)^2\}}\,dy$

$$=\int_{-x_1}^{x_1}\tfrac{1}{4}\pi c^{-1}A^{\frac{1}{2}}(y_2-y_1)^2\,dx=\pi\Delta A^{-\frac{3}{2}}\int_{-x_1}^{x_1}(x_1^2-x^2)\,dx=\tfrac{4}{3}\pi\Delta^{-\frac{1}{2}}.$$

(5) Let $x=ax'$, $y=by'$, $z=cz'$, then the volume is $abc\iiint dx'dy'dz'$, the limits being those determined by the equation

$$(x'^2+y'^2+z'^2)^2=x'^2+y'^2-z'^2,$$

or in polar coordinates $r^2 = 1 - 2\sin^2\theta$, hence the volume is

$$abc\int\int\int r^2 \cos\theta\, d\theta\, dr\, d\phi = \tfrac{2}{3}\pi abc\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \cos\theta\, d\theta\, (1 - 2\sin^2\theta)^{\frac{3}{2}};$$

$$\text{let } \sqrt{2}\sin\theta = \sin\psi,$$

$$\text{the volume} = \tfrac{2}{3}\pi abc\int_0^{\frac{1}{2}\pi} \sqrt{2}\cos^4\psi\, d\psi = \tfrac{1}{8}\sqrt{2}\pi^2 abc.$$

(6) Let θ be the angle which CP makes with the fixed diameter, the volume generated by the circle when its centre describes the arc $ad\theta$ is ultimately $\pi a^2 \sin^2\theta . ad\theta$, and the volume required is $4\int_0^{\frac{1}{2}\pi} \pi a^3 \sin^2\theta\, d\theta$.

(7) Let $ax + by = \xi$ be the equation of the line AB cutting the axes Ox, Oy in A, B; draw OY perpendicular to AB, and let η be the distance from Y of a point P in YA.

An element of the surface, whose projection on xy is the plane element at P, $= \sec\gamma\, d\eta\, d\xi / \sqrt{(a^2 + b^2)}$,

$$\text{where } \sec\gamma = \sqrt{\{1 + (a^2 + b^2)[f'(\xi)]^2\}}.$$

The result is the summation from $\eta = -BY$ to YA, $\xi = 0$ to c.

(8) Let p be the perpendicular from O on the tangent plane at P, the volume of the cone, whose vertex is O and base dS, is $\frac{1}{3}pdS$, hence the volume of the closed surface is $\frac{1}{3}\int\int r\cos\phi\, dS$.

Let the equation of the ellipsoid be $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$; the cosine of the inclination of dS to the plane of yz is px/a^2, $\therefore pdS = a^2 dy\, dz/x$, and if $y = br\cos\theta$, $z = cr\sin\theta$, $dy\, dz = bcr\, dr\, d\theta$, and $x = a\sqrt{(1 - r^2)}$, hence the volume is

$$\tfrac{8}{3}\int_0^1\int_0^{\frac{1}{2}\pi} abc . r\, dr\, d\theta / \sqrt{(1 - r^2)} = \tfrac{4}{3}\pi abc.$$

LXIX.

(1) Let the planes of the ellipses be $\alpha x + \beta y + \gamma z = \pm 1$,

$$\therefore \rho\,(ux^2 + \ldots + 2fyz + \ldots) \equiv (\alpha x + \beta y + \gamma z)^2 - x^2/a^2 - y^2/b^2 - z^2/c^2,$$

$$\therefore \alpha^2 - a^{-2} = \rho u,\ \ldots,\ \beta\gamma = \rho f,\ \gamma\alpha = \rho g,\ \alpha\beta = \rho h,$$

$$\therefore \alpha^2 = \rho gh/f,\ a^{-2} = \rho\,(gh/f - u),\ \&c. \quad (1)$$

Let $x = a\xi$, $y = b\eta$, $z = c\zeta$, the volume required $= \int\int\int dx\, dy\, dz$ $= abc\int\int\int d\xi\, d\eta\, d\zeta$, between proper limits, $= abc \times$ volume of the sphere $\xi^2 + \eta^2 + \zeta^2 = 1$ cut off by the cone whose vertex is the centre, and which intersects the sphere in planes $\alpha a\xi + \beta b\eta + \gamma c\zeta = \pm 1$.

Shew that this is $\frac{4}{3}\pi abc(1 - k)$, where $2k$ is the distance between the planes, so that $k^{-2} = \alpha^2 a^2 + \beta^2 b^2 + \gamma^2 c^2$,

$$\text{and, by (1), } \alpha^2 a^2 = gh/(gh - uf),\ \&c.$$

(2) Let $x = r\cos\theta$, $z = r\sin\theta$; then $(r - a)^2 + y^2 = (a\theta/2\pi)^2$. Hence the surface is generated by the motion of a variable circle, the plane of which turns round the axis of y, the centre describing

a circle of radius a, in the plane of zx, the radius being $a\theta/2\pi$, where θ is the angle through which the plane has revolved from the plane of xy. The volume required is $\int_0^{2\pi} a\,d\theta\,.\,a^2\theta^2/4\pi = \frac{2}{3}\pi^2 a^3$.

(3) The equation of the ellipsoid being $x^2/a^2+y^2/b^2+z^2/c^2=1$, if $y=b\sin\phi\cos\theta$, $z=c\sin\phi\sin\theta$, $x=a\cos\phi$,

$$\therefore \iint x^2 dS/p = \iint a^2 x\,dy\,dz\,(x^2/a^4+y^2/b^4+z^2/c^4)$$
$$=8\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} a^3 bc\sin\phi\cos^2\phi\,d\phi\,d\theta\,\{\cos^2\phi/a^2+(\cos^2\theta/b^2+\sin^2\theta/c^2)\sin^2\phi\}.$$

(4) Since the surface is closed, any straight line through O will meet the surface in an even number of points P_1, P_1'; P_2, P_2'; ...P_n, P_n'. Consider any pair of points P, P', and let O be a point in the line joining them, and with centre O and radius a describe a sphere; a slender cone whose vertex is O will cut from the surface the elements dS at P and dS' at P', and from the sphere the element $d\sigma$; let $OP=r$ and $OP'=r'$, $dS\cos\psi/r^2=d\sigma/a^2$ and $dS'\cos\psi'/r'^2=\mp d\sigma/a^2$, $-$ or $+$ as O is without or within the surface. Hence, summing for all the surface, $\iint \mu\,dS\cos\psi/r^2=0$, or $\mu/a^2\times$ the whole surface of the sphere, according as O is without or within the surface considered.

(5) Taking the coordinate axes as in Art. 525, let ρ be the radius of the generating circle of an intermediate anchor ring, $\rho^2=z^2+(r-c)^2$, the principal radii of curvature at a point P are ρ and $\rho r/(r-c)$, and $\rho_1^{-1}+\rho_2^{-1}$, of Art. 952, is $\rho^{-1}\{1+(r-c)/r\}$, hence S, the surface of the ring,

$$=\iiint dx\,dy\,dz\,\{1+(r-c)/r\}\,\{z^2+(r-c)^2\}^{-\frac{1}{2}};$$

let $x=r\cos\phi$, $y=r\sin\phi$, $\therefore dx\,dy=r\,dr\,d\phi$, and let $r-c=\rho\cos\theta$, $z=\rho\sin\theta$, $\therefore dr\,dz=\rho\,d\rho\,d\theta$, and $dx\,dy\,dz=r\rho\,d\rho\,d\theta\,d\phi$;

$$\therefore S=\int_0^a\int_0^{2\pi}\int_0^{2\pi} d\rho\,d\theta\,d\phi\,(2\rho\cos\theta+c)=4\pi^2 ac.$$

(6) Let $x=ax'$, $y=by'$, $z=cz'$, then the integral is

$$\iiint abc\,dx'\,dy'\,dz'\,e^{2\xi},$$

where ξ is the perpendicular from (x', y', z') on the plane $ax+by+cz=0$, (1), the limits being determined by the sphere $x'^2+y'^2+z'^2=1$; transform the axes so that $O\xi$ is perpendicular to the plane (1), (ξ, η, ζ) being the point (x', y', z'), the integral is $abc\iiint e^{2\xi}\,d\xi\,d\eta\,d\zeta=abc\int_{-1}^{+1}\pi\,(1-\xi^2)\,e^{2\xi}\,d\xi$, which gives the result.

(7) Let C be the centre of the sphere, a its radius, and let dS be a circular belt whose centre M is in CO, and radius PM; let $CO=c$, $OP=r$, and let θ be the angle between CP and CO produced, so that $r^2=a^2+c^2-2ac\cos\theta$, then

$$\iint f(r)\,dS=\int f(r)\,.\,2\pi a\sin\theta\,.\,a\,d\theta=2\pi a/c\int_{a-c}^{a+c} rf(r)\,dr$$
$$=2\pi a/c\,\{\phi(a+c)-\phi(a-c)\},$$

where $\phi'(r) = rf(r)$; this being independent of c, the differential coefficient vanishes, thence shew that, for all values of c,

$$(a+c)f'(a+c) + f(a+c) = (a-c)f'(a-c) + f(a-c),$$

$$\therefore rf'(r) + f(r) = A + B(r-a)^2 + C(r-a)^4 + \ldots,$$

where A, B, C, ... are constant;

hence $rf(r) = Ar + A' + \frac{1}{3}B(r-a)^3 + \frac{1}{5}C(r-a)^5 + \ldots$.

(8) The equation of the surface is found in Prob. XVIII. (14); and if $x = r\cos\theta$, $y = r\sin\theta$, then $z = (a^2 - b^2)\,c\sin\theta\cos\theta/ab$.

The volume is $4\iint r^2\,d\theta\,dz = 4\iint d\theta\,dz\,(1 + z^2/c^2)/(\cos^2\theta/a^2 + \sin^2\theta/b^2)$ from $z = 0$ to $abc\,(b^{-2} - a^{-2})\sin\theta\cos\theta$, and from $\theta = 0$ to $\frac{1}{2}\pi$, let $u = a^{-2}\cos^2\theta + b^{-2}\sin^2\theta$, $\therefore$ $(b^{-2} - a^{-2})^2\sin^2\theta\cos^2\theta = (b^{-2} - u)(u - a^{-2})$,

and the volume is $2abc\int_{a^{-2}}^{b^{-2}} du/u\,\{1 + \frac{1}{3}a^2b^2(b^{-2} - u)(u - a^{-2})\}$

$$= \tfrac{1}{3}abc\left\{4\log u + 2a^2b^2(b^{-2} + a^{-2})\,u - a^2b^2u^2\right\}_{a^{-2}}^{b^{-2}}$$

$$= \tfrac{1}{3}abc\left\{8\log(a/b) + a^2b^2(b^{-4} - a^{-4})\right\}.$$

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